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Cosmological redshift in conformal spacetimes. The Segal's model

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Abstract. *A general framework for analyzing the cosmological redshift in conformal spacetimes is developed here. Application to the cosmological model of I.E. Segal is made. Contrarily to Segal's claim, there seems to be no redshift in this model.*

1. INTRODUCTION

In the second decade of this century W.M. Slipher and other astronomers discovered a redshift in the spectral lines of most of the distant galaxies. This fact (later called «cosmological redshift») can be interpreted as a special relativistic Doppler effect which allows to assign to each galaxy a recession velocity with respect to the Earth. In 1929 E. Hubble found a more or less *linear* relation between this velocity and the galaxy-distance; «expansive» cosmological models were originally based on this experimental fact. Besides of the evidence in favour of expansive models, coming from the discovery of the cosmic microwave background radiation in 1968 and from the modern elementary particle physics, these

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models still provide the most satisfactory explanation of the cosmological redshift, so that the pair cosmological redshift $\longleftrightarrow$ expanding universe has deeply rooted as an intuitive idea in the presentday cosmology.

In 1976, after several previous papers, I.E. Segal published a book where he proposed an alternative cosmological model, characterized essentially for being a *conformal* and (in a certain sense) *non expansive* model, from which nevertheless Segal claims to predict a *cosmological redshift* depending *quadratically* on the distance to the observer.

In the Segal's model:

(i) Space-time is a four-dimensional smooth manifold (C^∞, Hausdorff, paracompact), connected and endowed with a notion of causality induced by a conformal structure of signature $(-+++)$. No Lorentzian metric compatible with the conformal structure is preferred «a priori» by physical reasons (1).

(ii) Every reference frame (C^∞ future directed vector field) induces a metric in the conformal space. It's postulated that, on a large scale, the average trajectories described by matter correspond to the integral curves of a certain frame whose associated metric (see later) has good symmetry properties, in particular, spatial isotropy, spatial homogeneity and temporal homogeneity (at least 7-parameter group of isometries). These postulates lead to a unique (essentially) cosmos at large scale, which is conformally diffeomorphic to $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$, where $[-\mathrm{d}t^2 \oplus h]$ indicates the conformal structure associated to the Lorentzian metric $-\mathrm{d}t^2 \oplus h$ (with $\mathrm{d}t^2$ and h the standard Riemannian structures on $\mathbb{R}$ and S^3 respectively).

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(1) A remark on conformal spacetimes should be made at this point. From the experimental local validity of the special relativity and of the inertia law, arguments can be given [1] to assume the following two ingredients for any reasonable macroscopic model of the spacetime (taken as a four dimensional, smooth, connected manifold) with freely falling massive and massless (light) test-particles on it:

a) A *conformal structure* (roughly speaking, a globally defined field of null cones) induced by equivalence classes (up to multiplication by smooth positive functions) of locally defined (pseudo) metrics of index 1 (Lorentzian type), lightpaths being (when suitably parametrized) null-geodesics of any of these Lorentzian metrics.

b) A (global) symmetric linear *connection*, whose parallel transport preserves nullity of vectors (and of course orthogonality) and for which all trajectories of (freely falling) test-particles are geodesics.

Now it is not clear at this point why the above connection should be metric, i.e. why a (mathematically always existing) global Lorentzian metric should be *physically relevant*; this assumption (whose experimental validation would involve parallel transport along paths beyond our domain of accessibility) appears as an «extraneous element of the theory» [1] (of course it can be very useful to make it, as e.g. in the Einstein's theory). Moreover, if we do not want to do any dynamics at all, but only to explain some «kinematical» effect (as the cosmological redshift appears to be) we do not need to speak about that connection, and so the assumption (b) can be dropped out.

(iii) If the charts where the analysis of the observational data is done (analysis which necessarily takes place in the Minkowski space of special relativity) were normal charts of the metric $-dt^2 \oplus h$, the observational appearance of the average trajectories of the matter would be «static» and there would not exist the possibility of explaining the cosmological redshift in this model. But Segal proposes an alternative way to assign coordinates, based on a certain conformal mapping between $(\mathbb{R} \times S^3, [-dt^2 \oplus h])$ and the Minkowski space, as the unique «anthropomorphically possible». It turns out that in these coordinates the above observational appearance is not already static. Segal claims that, due to this «apparent expansion», his model predicts a cosmological redshift and, when calculating it, he finds for small cosmological distances a square-law redshift--distance dependence (and not a linear dependence as observed by E. Hubble). Half of the Segal's book is devoted to show that a careful statistical analysis of the observational data favoures the square-law against the linear one.

The above mentioned extense experimental evidence in favour of (metrical) expansive cosmological models, and the mathematical complexity of the Segal's model have been probably the reasons why up to now that model has not received much attention from people studying cosmology (besides some articles of Segal and collaborators published after the mentioned book; see [12]). Recently R.M. Soneira ([11]) has casted doubts on the validity of the statistical analysis done by Segal to support his result about the redshift-distance square-law.

But, appart from the hypothetical *physical* relevance of Segal's cosmological model, *its interest would lie in the prediction of a cosmological redshift due to an expansion that is only observational.* The present work is dedicated to prove that *there is actually no cosmological redshift in the Segal's model.*

The sketch of this work is the following. In section 2 we present some basic definitions and standard results on conformal spacetimes (CST). In section 3 we introduce the notions of observer in a time-oriented CST, reference frame (RF), sychronizability and proper time synchronizability (some of the definitions have been merely adapted, to the conformal case, from the standard ones in the metric case). We give the results that a RF metrizes in an unique way its definition domain and that a proper time synchronizable RF is geodesic for the induced metric connection. We introduce the notions of factorizer reference frame (FRF), spatial homogeneity and isotropy and temporal homogeneity and we give the result that a temporally homogeneous FRF is parallel for the induced metric connection.

In section 4 questions related to the redshift in a CST are formalized, assuming that light-paths are (when suitably parametrized) nullgeodesics of any Lorentzian metric compatible with the conformal structure (what agrees with the scheme of a CST-based model (1)); theorem 1 establishes that the temporal homogeneity of a FRF implies that there is not redshift for the radiation emitted and received between any two of its observers.

Anticipating section 6 we want to make here the following remarks:

a) Segal's calculation to obtain a cosmological redshift in his model reduces to the calculation of the proper time ratio for a certain pair of observers. Segal obtains a value *not equal to one*; the calculation we propose as the correct one gives a value $r = 1$, and in that result we are based to affirm that such redshift does not exist.

b) Segal does his calculation in the framework of *quantum mechanics*. And as it does not seem (from a physical point of view) that an analysis of the cosmological redshift would need a quantum approach, our aim has been giving a *geometrical (classical) version* of Segal's calculation.

c) It could be thought that the definitions and results we have established as «classical analogues» of Segal's formalism have been given ad hoc in order to obtain a value $r = 1$ there where Segal obtains a value non equal to one. That is not the case; in fact we are going to see in section 6 that the formalism developed in section 4 (based essentially on the results of section 3) *allows to reproduce exactly the calculation of Segal*. The result by Segal – which, once suitably modified gives the value $r = 1$ – is independent of having followed either a quantum or a classical (geometrical) approach.

In section 5 we study the Segal's cosmological model, finding that the conformal mapping between $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ and the (metric) Minkowski space, which gives Segal's coordinatization of the cosmos by local observers:

1) agrees with the one given in [6], p. 118 - 121

2) defines charts which are normal charts, non of the global Lorentzian metric $-\mathrm{d}t^2 \oplus h$, but of local Lorentzian metrics, induced (in the sense of section 3) by certain locally defined temporally homogeneous FRF, whose explicit expression is also obtained.

In section 6 the possible existence of redshift in Segal's cosmological model is analyzed. *The answer is negative*, and this is an inmediate consequence of the results of section 4 and of the temporal homogeneity postulated by Segal for the reference system whose integral curves correspond to the average trajectories of matter. The rest of this section is devoted to analyze Segal's calculation in detail: our main disagreement can be summarized by saying that the redshift is not related to a «time evolution» but rather to a light propagation («spacetime evolution»).

We give in the Appendix the explict expression of the 15 fundamental fields associated to the action of the causal group of Segal's CST.

About notation: when the same index appears twice in a term, once as a subscript and once as a superscript, it will be understood a summation extended over the set of all possible values of that index, i.e.: $\{-1, 0, 1, 2, 3, 4\}$, for the indices $a, b, \ldots$, $\{0, 1, 2, 3\}$ for indices $\mu, \nu, \ldots$ and $\{1, 2, 3\}$ for indices $i, j, \ldots$.

2. CONFORMAL SPACETIMES

DEFINITION 1.

(i) Let L be a real finite-dimensional vector space. A *conformal linear structure* on L is an equivalence class C of nondegenerate symmetric bilinear forms on L, equivalence being defined as $b \sim b'$ if and only if $b' = ab$ for some a (real) > 0. We shall write: $[b] = [b']$. The pair (L, C) is called a *conformal linear space*. If $b \in C$, it is said that b *induces* the conformal structure C and that b and C are *compatible*.

(ii) Let M be a manifold. A *conformal structure* $\mathscr{C}$ on M is an assignement to each point $p \in M$ of a conformal linear structure $\mathscr{C}_p$ on T_pM with the following differentiability condition: $\forall p \in M$, $\exists$ neighborhood U of p and (semi Riemannian structure g, defined in U, such that $[g_q] = \mathscr{C}_q$, $\forall q \in U$.

$(M, \mathscr{C})$ is called a *conformal manifold* and it is said that g and $\mathscr{C}$ are compatible, what is represented by $g \in \mathscr{C}|_U$.

(iii) If M is a connected and four-dimensional manifold and if the above mentioned local metrics g are Lorentzian (i.e., they have index 1 or, equivalently, signature $(-+++)$), $(M, \mathscr{C})$ will be called a *conformal spacetime* (*CST*).

(iv) Let $(M, \mathscr{C})$ be CST, $p \in M$, $v \in T_pM$ and g a Lorentzian metric compatible with $\mathscr{C}$ and whose domain contains p. v is called:

- *timelike* if $g(v, v) < 0$
- *lightlike* if $v \neq 0$ and $g(v, v) = 0$
- *spacelike* if $v = 0$ or $g(v, v) > 0$.

These definitions (obviously independent of the selected g compatible with $\mathscr{C}$) allow to classify the vectors in TM in three disjoint subsets: timelike vectors ($\mathscr{T}$), lightlike vectors ($\mathscr{L}$) and spacelike vectors ($\mathscr{E}$). Then $TM = \mathscr{T} \cup \mathscr{L} \cup \mathscr{E}$.

The corresponding definitions are used for curves in M, vector fields on M, submanifolds of M, and so on.

The set $\mathscr{L}_p := \mathscr{L} \cap T_pM$ is called the *light cone in* p. Both $\mathscr{L}_p$ and $\mathscr{T}_p := \mathscr{T} \cap T_pM$ have two connected components. ([3], Exercise 1.1..9).

The existence of a conformal structure $\mathscr{C}$ on a manifold M restricts the manifold topology. In particular, there exists the following:

LEMMA 1. *Let M be a manifold of dimension $\geqslant 2$. Then, all the following statements are equivalent:*

(i) *M admits a conformal structure $\mathscr{C}$ induced by local semi-Riemannian* *structures with index 1.*

(ii) *M admits a (global) semi-Riemannian structure with index 1.*

(iii) *M admits a line field.*

(iv) *M admits a nowhere vanishing vector field.*

Proof.

(i) $\Rightarrow$ (ii) That is a consequence of a standard argument which uses partitions of unity and the local semi-Riemannian structures inducing $\mathscr{C}$.

(ii) $\Rightarrow$ (i) It is trivial.

(ii) $\Leftrightarrow$ (iii) See ([6]), p. 39 - 40); in that (constructive) proof the obtained [resp. assumed] line-field becomes timelike for the assumed [resp. obtained] semi-Riemannian structure.

(ii) $\Leftrightarrow$ (iv) See ([7], p. 149); the proof of $\Rightarrow$ is not constructive and the admissible vector field does not need to be timelike for the assumed semi-Riemannian structure. ■

It is well known that in a CST $(M, \mathscr{C})$ the set $\mathscr{T} \subset TM$ is an open submanifold with one or two connected components ([3], Propos. 1.2.1).

DEFINITION 2.

(i) A CST $(M, \mathscr{C})$ is said to be *time orientable* if $\mathscr{T}$ has two connected components. The choice of one of these components as $\mathscr{T}^+$ *(future)* and the other one as $\mathscr{T}^-$ *(past)* constitutes a *temporal orientation* and the CST is then said to be *time-oriented*.

(ii) Let $(M, \mathscr{C})$ be a time-oriented CST. A timelike vector $v \in \mathscr{T}_p$ (with $p \in$ $\in M$) is called *future-directed* (resp. *past-directed*) if $v \in \mathscr{T}_p^+ := \mathscr{T}^+ \cap T_pM$ (resp., $v \in \mathscr{T}_p^- := \mathscr{T}^- \cap T_pM$).

Given $v \in \mathscr{T}_p^+$, any future-directed timelike vector w (resp. past directed) at p satisfies $g(v, w) < 0$ (resp., $g(v, w) > 0$), g being compatible with $\mathscr{C}$. In addition, every lightlike vector n at p satisfies $g(v, n) < 0$ or $g(v, n) > 0$, $\forall\, v \in \mathscr{T}_p^+$, being then called *future-directed* or *past-directed* respectively ([3], 1.1.9).

The set $\mathscr{L}_p^+$ of future-directed lightlike vectors at p satisfies $\overline{\mathscr{T}}_p^+ = \mathscr{L}_p^+ \cup$ $\cup \{0\} \cup \mathscr{T}_p^+$ (and similarly for the set $\mathscr{L}_p^-$ of past-directed lightlike vectors). All these concepts can be applied in an obvious way to vector fields and curves.

Time-orientability is not only mathematically convenient, but also corresponds to our physical intuition which is based, for example, in the knowledge about thermodynamical processes (increment of entropy) here and now on the Earth. Thus, it is reasonable to assume the existence of a smooth field of timelike vectors defined everywhere on the spacetime.

LEMMA 2.

(i) *A CST is time-orientable if and only if it admits a global timelike* C^∞ *vector field. Once a time-orientation has been chosen the field is either future-directed at every point or past-directed at every point.*

(ii) *A simply connected CST is time-orientable.*

Proof. See [4], p. 17, 19. ■

REMARK. Let M be a manifold with dimension $\geqslant 2$. Then each one of the statements (i) to (iv) from Lemma 1 is equivalent to:

(v) M admits a *timelike* vector field *for some of* the admisible globally defined semi-Riemannian structures with index 1.

This can be expressed in other way by saying that any manifold which admits a CST structure, admits also a time-oriented CST structure. ([7], p. 149).

Time-orientability is not the only condition which can be reasonably imposed on a CST. It is usually assumed also the so-called *causality condition*, which implies the non existence of closed non-spacelike future-directed curves (this corresponds to the idea that no event can be «caused by itself»). Other more restrictive conditions (like strong causality, stable causality and global hyperbolicity), which are satisfied by classical cosmological models and by the Segal model, will not be discussed here.

DEFINITION 3. Let $(M, \mathscr{C})$ be a time-oriented CST. The *causal group* of $(M, \mathscr{C})$ is defined as the group of all conformal diffeomorphisms of $(M, \mathscr{C})$ onto itself which preserve the time-orientation.

LEMMA 3. *Let* $(M, \mathscr{C})$ *be a time-oriented CST. Then, its causal group is a Lie group of dimension equal or less that 15.*

Proof. See [5], example 1.2.6 and theorem 1.5.1.

It is clear that if two time-oriented CST are causally diffeomorphic, then their causal groups are isomorphic as Lie groups. ■

As with Killing fields in the metric case, we have the following:

DEFINITION 4. Let $(M, \mathscr{C})$ be a time-oriented CST. A vector field X on M is *causal* if the local one-parameter group of local diffeomorphisms generated by X

is causal. i.e., if and only if the following holds: $\forall g \in \mathscr{C}$ with non-empty $\mathrm{Dom}\, X \cap \mathrm{Dom}\, g$, $L_X g = \sigma g$ (σ being a certain C^∞ function defined on the intersection of the domains of X and g).

The maximal number of R-linearly independent causal fields on any open subset of M will be equal or less than 15 (see Lemma 3).

We shall mention, at last, a result ([2], p. 26 and 39) about the posibility of having time-oriented conformal spacetimes with identical local structure, but with different global topology.

LEMMA 4. *Let $(M, \mathscr{C})$ be a CST. If $\Phi : \widetilde{M} \to M$ is a covering of M, then $(\widetilde{M}, \Phi^*\mathscr{C})$ is a CST, defining $\Phi^*\mathscr{C} := [\Phi^* g]$ (g being any global metric on M compatible with $\mathscr{C}$, see Lemma 1). If $(M, \mathscr{C})$ is time-oriented, $(\widetilde{M}, \Phi^*\mathscr{C})$ is also time-oriented.*

Example 1. Given any four-dimensional connected manifold endowed with a Lorentzian metric, a CST can be obtained by considering the conformal structure associated to that metric.

Minkowski space is the spacetime of special relativity. It is based on the manifold $\mathbb{R}^4$ with metric $\eta = -\,dx^0 \otimes dx^0 + \sum_{i=1}^{3} dx^i \otimes dx^i$ (x^0, x^1, x^2, x^3 being the usual coordinate functions in $\mathbb{R}^4$).

We denote by $(\mathbb{R}^4, [\eta])$ the associated CST. The globally defined field $\partial/\partial x^0$ is a timelike vectorfield, so $(\mathbb{R}^4, [\eta])$ is time-orientable. Defining $\partial/\partial x^0$ as «future», $(\mathbb{R}^4, [\eta])$ becomes time-oriented and also satisfies the causality-condition.

The number of $\mathbb{R}$-linearly independent causal fields is 15, i.e., maximal, but four of these fields are not complete; in fact, it can be easily seen that the four following fields:

$$X_0 := \frac{\eta(x, x)}{2}\, \partial_{x^0} + x^0 S$$

and

$$X_i := \frac{\eta(x, x)}{2}\, \partial_{x^i} - x^i S \qquad (i = 1, 2, 3), \quad \text{with}$$

$$S \equiv x^\nu \partial_{x^\nu} \quad (\nu = 0, 1, 2, 3)$$

are causal fields (satisfying $L_{X_\nu} \eta = 2 x^\nu \eta$, $\nu = 0, 1, 2, 3$), and nevertheless their integral curves are not defined on the whole $\mathbb{R}$, i.e., they do not generate elements of the causal group of $(\mathbb{R}^4, [\eta])$ with, in fact, has dimension 11 ([8]); ten of those dimensions correspond to the isometries of η (Poincaré group or inhomogeneous Lorentz group– and the other one corresponds to the scale transformation generated by the (complete) field S. It is remarkable that this casual group acts in an affine way on R^4 ([8]).

Example 2. $S^1 \times S^3$, with $S^1 := \{(u_{-1}, u_0) \in \mathbb{R}^2 : u_{-1}^2 + u_0^2 = 1\}$ and $S^3 := \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1\}$ (with the usual coordinates in $\mathbb{R}^2$ and $\mathbb{R}^4$ respectively), is a four-dimensional connected manifold. If we denote by l and h the standard Riemannian structures defined on the spheres S^1 and S^3, then $g := -l \oplus h$ is a Lorentzian metric on $S^1 \times S^3$. In this way is obtained the CST $(S^1 \times S^3, [g])$.

The map $\alpha : \mathbb{R} \to S^1$, $s \mapsto (\cos s, \sin s)$ is a local diffeomorphism and takes the standard vector field on $\mathbb{R}$ onto a global vector field X on S^1. Taking in the obvious way X as a vector field on $S^1 \times S^3$, X is a globally defined timelike field (it is easy to show that $g(X, X) = -1$). Chosing X as «future», $S^1 \times S^3$ becomes time-oriented.

This CST does not satisfy the causality condition. Nevertheless, if we consider the universal covering $\Psi : \mathbb{R} \times S^3 \to S^1 \times S^3$, $(s, p) \mapsto (\alpha(s), p)$, the CST $(\mathbb{R} \times S^3, [\Psi^* g])$ becomes time-oriented and satisfies the causality condition, (We have $\Psi^* g = -\mathrm{d}t^2 \oplus h$).

3. OBSERVERS AND REFERENCE FRAMES

An observer is roughly speaking a person who identifies the events of his history with the reading of his clock; inevitably an observer «moves» through space-time towards the future. The simplest mathematical representation of an observer is given by a timelike future-directed curve, whose trajectory represents all his history, and whose parameter represents the observer's notion of time.

DEFINITION 5. Let $(M, \mathscr{C})$ be a time-oriented CST. An *observer* on M is a timelike future-directed curve $\gamma : \mathbb{R} \supset \Delta \to M$, whose parametrization defines the observer's *proper time*.

Any reparametrization of the curve γ represents a change in the observer's way of measuring time and, in fact, a change of observer; if a person chooses a certain physical process easily reproducible along his trajectory in the spacetime as a clock (for example, some atomic transition occurring in a piece of matter he brings along), he is parametrizing, in an univoque way, his trajectory, and so defining an observer.

Strictly speaking, an observer can only register events which take place along his trajectory in the space-time. To obtain information about any other events he needs the cooperation of other observers, who, in addition to himself constitute a reference frame.

DEFINITION 6. Let $(M, \mathscr{C})$ be a time-oriented CST. A *reference frame* (RF) in M is a future-directed timelike vector field Q defined on an open subset $U \subset M$. Each one of the integral curves of Q will be called an *observer of Q*. The reference frame is *global* if $U = M$; in any other case is *local*.

A corollary of Lemma 2 is the following

LEMMA 5. *A time-oriented CST admits a global reference frame.*

To be operative a reference frame Q must provide a notion of «simultaneity», what implies that the region U where Q is defined can be «sliced up» by three--dimensional spatial submanifolds, orthogonal to every observer in Q, and being «level hypersurfaces» of a certain time-function t defined on U: in that case Q will be called synchronizable. If the time function coincides, for every observer of Q, with the proper time of the observer, Q is called proper time synchronizable. In general, this coincidence will not take place; neverheless the observers of Q have an explicit way to compute, from the read out of their clocks (and assuming that the hour of all these clocks is adjusted to a standard value on a certain surface of simultaneity) which one is at every moment the surface of simultaneity they are «crossing». We are going to precise these definitions right now.

DEFINITION 7. Let $(M, \mathscr{C})$ be a time-oriented CST. A RF Q with domain $U \subset M$ is *synchronizable* if, given $g \in \mathscr{C}|_U$, we can find functions h, t on U, with $h > 0$ and such that $g(Q, .) = -h\,dt$. We will say that t is a *time function* for Q.

Note that if the condition of synchronizability is fulfilled for some $g \in \mathscr{C}|_U$, it will be fulfilled also for any other $g' \in \mathscr{C}|_U$. The following Lemma justifies the terminology used in this definition.

LEMMA 6. *Let $(M, \mathscr{C})$ be a time-oriented CST and let Q be a synchronisable reference frame with domain $U \subset M$. Let t be a time function for Q. Then at each point $m \in U$ there exists one unique three-dimensional submanifold orthogonal to Q (and thus spatial) and maximal in U. This submanifold corresponds to the hypersurface $t = t(m)$.*

Proof. It is straightforward, taking into account that $\underset{\sim}{Q} \wedge d\underset{\sim}{Q} = 0$ ($\underset{\sim}{Q}$ being the 1-form $g(Q, .)$) and applying the Froubenius theorem. ■

DEFINITION 8. Let $(M,\mathscr{C})$ be a time-oriented CST. A synchronizable reference frame in M, Q, is said to be *proper time synchronizable* if there exists some time function t for Q such that $Q(t) = 1$,. We will call t (now unique up to an additive constant) the *proper time function for Q*.

LEMMA 7. *Let $(M,\mathscr{C})$ be a time-oriented CST and let Q be a proper time synchronizable reference frame with domain $U \subset M$, t being the proper time function for Q. Then t coincides with the proper time of each one of the observers of Q (up to an additive constant).*

The proof of this Lemma is trivial. ∎

The following Lemma (whose proof is also trivial) establishes that a reference frame endows in an unique way a metric the region of the CST in which it is defined, so that the proper time of each one of its observers coincides with the arc-lenght measured along the corresponding trajectory.

LEMMA 8. *Let $(M,\mathscr{C})$ be a time-oriented CST and let Q be a reference frame with domain $U \subset M$. Then, there exists an unique Lorentzian metric $g^Q \in \mathscr{C}\restriction_U$ defined on U such that $g^Q(Q, Q) = -1$.*

LEMMA 9. *Let $(M,\mathscr{C})$ be a time-oriented CST and let Q be a reference frame with domain $U \subset M$. Then Q is proper time synchronizable if and only if there exists a function t on U such that $g^Q(Q, .) = -\mathrm{d}t$.*

The proof is trivial. ∎

Now we are going to deduce some properties of the reference frames related to their associated metric connections. If Q is a reference frame, we will denote by ∇^Q the Levi-Civita connection corresponding to g^Q. Firstly note that the condition $g^Q(Q, Q) =$ constant is necessary but not sufficient for Q to be a geodesic field of ∇^Q. One of the properties of a proper time synchronizable RF is to be geodesic for the induced metric connection:

LEMMA 10. *Let $(M,\mathscr{C})$ be a time-oriented CST and let Q be a proper time synchronizable reference frame of M. Then $\nabla^Q_Q Q = 0$.*

Proof. The proof is a straightforward consequence of the fact that ∇^Q satisfies the Ricci-identity and has zero torsion. ∎

Some proper time synchronizable reference frames «factorize» their domain in spatial and temporal parts, preserving the conformal structure, as we precise in the following:

DEFINITION 9. Let $(M, \mathscr{C})$ be a time-oriented CST. A proper time synchronizable RF Q with domain $U \subset M$ is said to be a *factorizer (FRF) of U* if there exists a diffeomorphism $\Phi : T \times S \to U$, being T a certain open interval of $\mathbb{R}$ and S a certain three-dimensional manifold, such that:

(i) It holds $\Phi_* \partial_t = Q$, ∂_t being the standard field on T considered as vector field on $T \times S$.

(ii) There exist Riemannian structures l and h on T and S, respectively, so that Φ is a causal diffeomorphism from $(T \times S, [-l \oplus h])$ onto $(U, \mathscr{C}|_U)$.

That definition of FRF corresponds to the notion of «metric observer» given by Segal ([2]) and includes the «comoving frames» of the classical cosmological models.

The properties of factorizers are summarized in the following:

LEMMA 11. *Let $(M, \mathscr{C})$ be a time-oriented CST, and let Q be a FRF of $U \subset M$. Using notation from definition 9 we have:*

(a) $\Phi^* g^Q = -\mathrm{d}t^2 \oplus fh$, *$f$ being a certain (positive-definite) function on the first factor in $T \times S$. Therefore, $t \circ \Phi^{-1}$ is a proper time function for Q. (We denote by t both the identity chart t and the mapping $t \circ \Pi_1$, with $\Pi_1 : T \times S \to T$ the canonical projection).*

(b) *The following statements are equivalent:*

- (i) *Translations on T, $\alpha : T \times S \to T \times S$, $(t, s) \to (t + a, s)$ (with t, $t + a \in T$) are causal on $(T \times S, [-l \oplus h])$.*
- (ii) *translations on T are isometries of the metric $\Phi^* g^Q$.*
- (iii) *the function f from* (a) *is constant.*

Proof.

(a) Obviously $\Phi^* g^Q = \Omega(-l \oplus h) = \Omega\, \psi\left(-\mathrm{d}t^2 \oplus \dfrac{1}{\psi}\, h\right)$, Ω (resp. ψ) being a positive-definite function on $T \times S$ (resp. on T).

Buth: $-1 = g^Q(Q, Q) = (\Phi^* g^Q)(\partial_t, \partial_t) = -\Omega\psi$; thus $\Phi^* g^Q = -\mathrm{d}t^2 \oplus fh$, with $f \equiv 1/\psi$.

(b) Because of $L_{\partial_t}(\Phi^* g^Q) = L_{\partial_t}(-\mathrm{d}t^2 \oplus fh) = 0 \oplus (\partial_t f) h$, we will have:

(i) $\Leftrightarrow$ (ii) Assume $(\forall g \in [-l \oplus h])$ $L_{\partial_t} g = \sigma_g g$, for some function σ_g; taking $g \equiv \Phi^* g^Q$, the above remark leads to $\sigma_g = 0$, thus

$L_{\partial_t}(\Phi^* g^Q) = 0$. Conversely, if $L_{\partial_t}(\Phi^* g^Q) = 0$, we will have: $L_{\partial_t} g \equiv L_{\partial_t}(f'\Phi^* g^Q) = \sigma_g g$, with $\sigma_g = (1/f')(\partial_t f'), f' > 0$.

(ii) ⇔ (iii) (trivial). ■

In classical cosmological models the «comoving frames» are factorizers. This property comes from the proper time synchronizability of these frames and from the assumptions (justified by the so-called «Cosmological Principle» and by the observational data, see [3] and [10]) of spatial homogeneity and isotropy (see below).

DEFINITION 10. Let $(M,\mathscr{C})$ be a time-oriented CST. Let Q be a FRF of U (as in definition 9).

(a) Q is *temporally homogeneous* if any of the statements from Lemma 11 (b) holds.

(b) Q is *spatially homogeneous* if (S, h) has a three-dimensional transitive group of isometries.

(c) Q is *spatially isotropic* at a point $s \in S$ if (S, h) has a three dimensional group of isometries which keep s fixed.

Note that spatial isotropy at any point implies spatial homogeneity and, viceversa, spatial homogeneity and isotropy at one point imply isotropy at any other point. It is easy to prove the following lemma, which establishes the natural correspondence among isometries in (T, l) or in (S, h) (both defined by a certain FRF) and causal diffeomorphisms.

LEMMA 12. *Let $(M,\mathscr{C})$ be a time-oriented CST and let Q be a FRF of $U \subset M$. Using notation from definition 9, we define:*

$$\chi : \mathrm{Dif}(S) \longrightarrow \mathrm{Dif}(U)$$

$$a \longrightarrow \chi(a), \quad \textit{with} \quad \chi(a)\,\Phi(t,s) := \Phi(t, as)$$

and

$$\xi : \mathrm{Dif}(T) \longrightarrow \mathrm{Dif}(U)$$

$$b \longrightarrow \xi(b), \quad \textit{with} \quad \xi(b)\,\Phi(t,s) := \Phi(bt, s)$$

being Dif( ) *the set of diffeomorphisms of the manifold onto itself.*

Then it follows that a [resp. b] is an isometry of (S, h) [resp. (T, l)] iff $\chi(a)$ [resp. $\xi(b)$] is causal in $(U, \mathscr{C}|_U)$ iff $\chi(a)$ [resp. $\xi(b)$] is an isometry of (U, g^Q).

It follows from the foregoing Lemma that if a FRF satisfies the assumptions of definition 10(b) and (c), then the metric manifold (U, g^Q) will have a (at least) six-dimensional group of isometries.

It the FRF is also temporally homogeneous, then the group will be at least seven-dimensional. Note that the maximum dimension of the group of isometries is ten ([7]).

In the Lemma 10 it has been shown that a proper time synchronizable RF Q is *geodesic* for ∇^Q. The following Lemma shows that if Q is also factorizer and temporally homogeneous, then it is *parallel* for ∇^Q.

LEMMA 13. *Let $(M,\mathscr{C})$ be a time oriented CST and let Q be a temporally homogeneous FRF with domain $U \subset M$. Then $\nabla^Q_X Q = 0$, for every field X defined on U.*

Proof. Q being a FRF, it follows from Lemma 11(a) that in a neighborhood of any point of U coordinates (x^0, x^1, x^2, x^3) exist such that

$$g^Q = -\,\mathrm{d}x^0 \otimes \mathrm{d}x^0 + f(x^0) h_{ij}(x^1, x^2, x^3)\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad i = 1, 2, 3,$$

and

$$Q \equiv \frac{\partial}{\partial x^0}.$$

In such coordinates, the coefficients of the metric connection ∇^Q are:

$$\Gamma^\alpha_{00} = \Gamma^0_{0\alpha} = 0 \quad \text{with} \quad \alpha = 0, 1, 2, 3$$

$$\Gamma^i_{0j} = 1/2\, \delta^i_j \dot{f}/f$$

$$\Gamma^0_{ij} = 1/2 \dot{f} h_{ij}$$

$$\Gamma^i_{jk} = 1/2\, h^{il}(h_{jl,k} + h_{kl,j} - h_{jk,l}) \quad \text{with} \qquad i, j, k, l = 1, 2, 3.$$

Let X be a field on U, with coordinates $X = X^\alpha\, \partial/\partial x^\alpha$,

$$\nabla^Q_X = X^\alpha \Gamma^\beta_{\alpha 0} \partial/\partial x^\beta = X^i \Gamma^j_{i0} \partial/\partial x^j = X^i \,\frac{\dot{f}}{2f}\, \delta^j_i\, \partial/\partial x^j = \frac{\dot{f}}{2f} X^i\, \partial/\partial x^i$$

As Q is temporally homogeneous, $\dot{f} = 0$ (Lemma 11(b)); thus $\nabla^Q_X Q = 0$. ■

Finally, and before going to the examples, we must take into account that a causal diffeomorphism maps a reference frame into another one with identical properties of synchronizability and symmetry:

LEMMA 14. *Let $(M,\mathscr{C})$ and $(M', \mathscr{C}')$ be two time-oriented CST. Let Φ be a* *causal diffeomorphism of M onto M', and let Q be a reference frame in M. Then $\Phi_* Q$ is a reference frame in M' and each one of the following properties is true for $\Phi_* Q$ if it is true for Q:*

(i) *being synchronizable*
(ii) *being proper time synchronizable*
(iii) *being factorizer*
(iv) *being temporally homogeneous*
(v) *being spatially homogeneous*
(vi) *being spatially isotropic at a certain point.*

Proving this lemma is only a mere question of calculus. We will two reference frames Q, Q' in M *conjugate* if there exists a causal diffeomorphism Φ such that $\Phi_* Q = Q'$.

Example 3. In the conformal version of the Minkowski spacetime $(\mathbb{R}^4, [\eta])$ (see example 1), the vector field $\partial/\partial x^0$ is a global (and complete) reference frame, with $g^{\partial/\partial x^0} = \eta$, and proper time synchronizable, the proper time being the coordinate x^0. Moreover it is a factorizer of $\mathbb{R}^4$, temporally homogeneous, spatially homogeneous and spatially isotropic. There are no other known global *complete* factorizers in $(\mathbb{R}^4, [\eta])$ with the same properties of symmetry but nonconjugate to $\partial/\partial x^0$ ([2] p. 46). We will call *inertial* those RF on $(\mathbb{R}^4, [\eta])$ which are conjugate to $\partial/\partial x^0$ by elements of the group of isometries of η (10-dimensional Poincaré group).

Example 4. Let $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ be as in example 2. Let Q be the field ∂_t on $\mathbb{R}$ considered as a vector field on $\mathbb{R} \times S^3$. It is straightforward to check that Q is a global (and complete) reference frame, with $g^a = -\mathrm{d}t^2 \oplus h$, and proper time synchronizable, with t as proper time. Furthermore Q is a factorizer of $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$, temporally homogeneous and spatially homogeneous and isotropic; in spherical coordinates $(\rho, \vartheta, \varphi)$ (see Appendix) the expression of h is $h = \mathrm{d}\rho \otimes \mathrm{d}\rho + \sin^2\rho\,(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi)$. There are no other known global *complete* factorizers in $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ with the same properties of symmetry but non conjugate to Q ([2]. p. 48).

4. RED SHIFTS

Let us assume that, in a piece of matter comoving with a certain observer γ, a particular and fixed atomic transition with emission of light takes place. During an interval of proper time $\Delta\tau$, n pulses of light are emitted (2); the frequency of the emission, observed by γ, will be, then,

$$\nu_{E|\gamma} = \frac{n}{\Delta\tau} .$$

Let us assume now that the n pulses are received by another observer γ', during an interval $\Delta\tau'$ of his proper time; the frequency of reception observed by γ' will be $\nu_{R|\gamma'} = \dfrac{n}{\Delta\tau'}$ and the ratio between both frequencies $r := \dfrac{\nu_{E|\gamma}}{\nu_{R|\gamma'}} = \dfrac{\Delta\tau'}{\Delta\tau}$. $r > 1$ (resp. $r < 1$) means that the emitted frequency, observed by γ relative to his proper time, is greater (resp. smaller) than the received frequency observed by γ' relative to his proper time. It is said then that γ' observes a *red shift* (resp. a blue shift) relative to the observations made by γ.

In astrophysical measurements the emitter is a star or galaxy and the receiver is an observer on Earth. But, as direct observation just *where the light is emitted* is impossible, the frequency of the received light is compared with the frequency of the light emitted on Earth by *the same light source* as in the galaxy. To interpret the corresponding frequency ratio it is assumed (physical hypothesis) that the proper time interval taken by the emission of a pulse is the same for every observer, provided that light is produced by the same physical process and the clocks which measure proper times are physically identical. This is a reasonable hypothesis, because it is expected that different positions in spacetime will eventually affect in the same way the process of light emission and the clock which measures time.

In some cases, a value $r \neq 1$ for the frequency ratio can be interpreted as a Doppler effect (see example 5) caused by the relative motion between the light source and the observatory on Earth (this is e.g. the case for signals coming from artificial satellites, as studied by special relativity in the absence of gravitation; see [10], §2.2, and [3], §5.4.2), or as due to the gravitational field along the light path (for example, in the case of signals propagating through strong stationary gravitational fields, as studied by the Einstein theory; see [10], §3.5, and [3], §7.2 and §7.4.3). Nevertheless it does not seem that the systematic observation of values $r > 1$ for the spectral lines coming from most of galaxies and quasars (the so-called *cosmological red shift*) can be interpreted in the above-

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(2) It does not seem that an analysis of the cosmological red shift would need a quantum approach to the radiation. In fact, the quantum treatment of the Segal's model has, in what concerns to the red shift, an immediate classical geometrical analogue

-mentioned way; the most generalized interpretation is still based on the so--called «expansive» cosmological models ([10], chapters 14 and 15; also example 6).

As we are going to see in brief, a time-oriented CST $(M,\mathscr{C})$ provides an appropriate framework to analyze the cosmological red shift, provided that light are assumed to be (when suitably parametrized) null-geodesics of any Lorentzian metric compatible with the conformal structure (see section 1). It is also necessary to postulate the cosmological relevance of a global FRF Q. Specifically, any of the so-called «typical» galaxies (i.e., whose trajectories are approximately those of the mean matter, see [10], chapter 14; our galaxy can be considered as typical, see [10], p. 110) will move along a world line described by an integral curve γ of the postulated FRF Q; and the parametrization of such curves will represent the ones obtained along the corresponding world lines with physically identical clocks comoving with each one of these galaxies.

DEFINITION 11. Let $(M,\mathscr{C})$ be a time-oriented CST and let $\gamma : \epsilon \to M$, $\gamma' : \epsilon' \to M$ be observers.

Assume that there exists a bijective function C^∞ $f : [c,d] \to [c',d']$ (with $[c,d] \subset \epsilon$, $[c',d'] \subset \epsilon'$) such that $\forall \tau \in [c,d]$, $\exists$ a lightlike future-directed curve between $\gamma\tau$ and $\gamma' f\tau$ which is geodesic for some $g \in \mathscr{C}$.

In this case we define the proper time *ratio r between* γ *and* γ' *at* $\gamma' f\tau_0$ (with $\tau_0 \in [c,d]$) as:

$$r := \left.\frac{df}{d\tau}\right|_{\tau_0} .$$

Remarks

(i) It is known that, given $g, g' \in \mathscr{C}$, any lightlike geodesic of g can be suitably reparametrized to be lightlike geodesic of g' (see [3], p. 132), the foregoing definition being thus *independent of the chosen g*.

(ii) As mentioned above lightike future-directed curves which represent light trajectories are those which allow reparametrizations making them geodesics of any $g \in \mathscr{C}$. Thus it is guaranted in the foregoing definition that *there exists some light path between* $\gamma\tau$ *and* $\gamma' f\tau$, $\forall \tau \in [c,d]$.

The *uniqueness* of this trajectory is not guaranted (nor, hence, the uniqueness of f), unless some additional assumption is made about «simple convexity» of certain open set containing $\gamma([c,d])$, $\gamma'([c',d'])$ and the ranges of all corresponding null geodesics.

(iii) The above-mentioned proper time ratio r *depnds* only on γ, γ', τ^0 and on the available f and *represents* the infinitesimal version of the ratio

$\dfrac{\Delta\tau'}{\Delta\tau}$; if $r > 1$, the receiver will observe a red shift. (In cosmology it is commonly used the «red shift index» $z := r - 1$).

The following lemma provides a method to compute r as a ratio between scalar products.

LEMMA 15. *Let $(M, \mathscr{C})$ be a time-oriented CST. Let $\gamma, \gamma', f, \tau_0$ be as in definition 11. Let be $g \in \mathscr{C}$ and $\lambda : [a, b] \to M$, with $\lambda a = \gamma\tau_0$, $\lambda b = \gamma' f \tau_0$, the lightlike future-directed geodesic of g whose existence is assumed in the definition 11. Then it holds:*

$$r = \frac{g(\lambda_* a, \gamma_* \tau_0)}{g(\lambda_* b, \gamma'_*(f\tau_0))} .$$

Proof. Let $\lambda^\tau : [a, b] \to M$, with $\lambda^\tau a = \gamma\tau$, $\lambda^\tau b = \gamma' f\tau$, be the corresponding lightlike future-directed geodesic between $\gamma\tau$ and $\gamma' f\tau$, $\forall \tau \in [c, d]$ (with $\lambda^{\tau_0} = \lambda$). Let be $\mathscr{D} := [a, b] \times [c, d] \subset \mathbb{R}^2$ and $\sigma : \mathscr{D} \to M$ defined by $\sigma(u, \tau) := \lambda^\tau u$. Then it holds $g(\sigma_* \partial/\partial x^1, \sigma_* \partial/\partial x^1) = 0$ (constant); if follows (Gauss lemma, see [3], §5.0.3) that $g(\sigma_* \partial/\partial x^1, \sigma_* \partial/\partial x^2)$ is constant along each geodesic λ^τ(x^1, x^2, standard coordinates in $\mathbb{R}^2$).

So, we can write:

$$g(\lambda_* a, \gamma_* \tau_0) = g(\lambda_* b, (\gamma'_0 f)_* \tau_0) = g\left(\lambda_* b, \gamma'_*(f\tau_0) \cdot \left.\frac{df}{d\tau}\right|_{\tau_0}\right),$$

and we obtain

$$r := \left.\frac{df}{d\tau}\right|_{\tau_0} = \frac{g(\lambda_* a, \gamma_* \tau_0)}{g(\lambda_* b, \gamma'_* f\tau_0)} . \qquad \blacksquare$$

Remarks

(i) The resulting value of r is not changed if another $g' \in \mathscr{C}$ instead of g is chosen (what is reasonable, because r is independent of g). In fact what happens is that the contribution from the conformal factor α (with $g' = \alpha g$) on both curves γ and γ' cancels the contribution coming from the reparametrization of the curve λ to maintain its geodesic character while changing g by g' (specifically, the new curve $\Lambda = \lambda \circ \beta$, with $\beta : [A, B] \to [a, b]$, satisfies $\Lambda_* u = \lambda_* \beta u \cdot \dfrac{d\beta}{du} = \dfrac{1}{\alpha(\lambda\beta u)} \lambda_* \beta u$).

(ii) The above mentioned dependence of r on γ, γ' and τ_0 is actually a dependence on the tangent vectors $\gamma_* \tau_0$ and $\gamma'_* f \tau_0$.

COROLLARY. *With the assumptions of Lemma 15 the following holds:*

$$r = \frac{g(\lambda_* a, \gamma_* \tau_0)}{g(\lambda_* a, \Pi^g_{b,a} \gamma'_* (f \tau_0))}$$

being $\Pi^g_{b,a}$ *the parallel transport defined by the Levi-Civita connection,* ∇^g, *along the curve* λ *between the points* $\lambda b = \gamma' f \tau_0$ *and* $\lambda a = \gamma \tau_0$.

Proof. It follows inmediately from the facts that Π^g preserves the scalar products defined by g and that λ is geodesic of ∇^g. ■

The next theorem establishes that between two observes of any temporally homogeneous FRF (not necessarily global) there is no red shift.

THEOREM 1. *Let* $(M, \mathscr{C})$ *be a time-oriented CST. Let* Q *be a temporally homogeneous FRF. Then for any pair of observers of* Q, γ *and* γ', *assuming the conditions from definition 4.1, the proper time ratio is equal to 1.*

Proof. In the corollary of Lemma 15, let us take $g :\Rightarrow g^Q$. As Q is temporally homogeneous FRF, it holds (Lemma 13) $\nabla^Q_v Q = 0$, $\forall v \in TM$ and, because $\nabla^g = \nabla^Q$, it follows that $\Pi^g_{b,a}(\gamma'_* f \tau_0) \equiv \Pi^g_{b,a}(Q(\lambda b)) = Q(\lambda a) \equiv \gamma_* \tau_0$. Thus $r = 1$. ■

Example 5. Let $(\mathbb{R}^4, [\eta])$ be the conformal version of Minkowski space (example 1). Let us consider two conjugate inertial reference frames, $\mathscr{T}$ and $\mathscr{G}$ (example 3). We are going to describe the so-called «relativistic Doppler effect».

Assume without loosing generality that $\mathscr{T} = \partial/\partial x^0$ and $\mathscr{G} = \delta \partial/\partial x^0 + \delta\beta \partial/\partial x^3$, with $-1 < \beta < 1$ and $\delta = (1 - \beta^2)^{-1/2}$. We say that observers of $\mathscr{G}$ move with velocity βc relative to $\mathscr{T}$ (c being the speed of light) in the direction of the x^3 coordinate. It is clear that $\mathscr{G}$ can be obtained from $\mathscr{T}$ by means of the Lorentz transformation $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$, $\Phi(v) = Av$, with

$$A = \begin{pmatrix} \delta & 0 & 0 & \delta\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \delta\beta & 0 & 0 & \delta \end{pmatrix}$$

Let us choose the following observer of $\mathscr{T}$, $\gamma' : \mathbb{R} \to \mathbb{R}^4$, $\gamma' \tau = (\tau, 0, 0, 0)$ as the light receiver, and some observer γ of $\mathcal{G}$ as the emitter, Our aim is to compute the proper time ratio r between γ and γ' at $\gamma' f 0$. Taking into account:

(a) $\mathcal{C}$ and $\mathcal{G}$ are conjugate inertial RF, so

$$g^{\mathcal{C}} = g^{\mathcal{G}} = \eta \qquad \nabla^{\mathcal{C}} = \nabla^{\mathcal{G}} = \nabla^{\eta}$$

(b) $\mathcal{C}$ and $\mathcal{G}$ are temporally homogeneous FRF, thus for every curve $\alpha : [a, b] \to \mathbb{R}^4$, it holds (Lemma 13):

$$\Pi^{g^{\mathcal{C}}}_{b,a}(\mathcal{C}(\alpha b)) = \mathcal{C}(\alpha a)$$

and

$$\Pi^{g^{\mathcal{G}}}_{b,a}(\mathcal{G}(\alpha b)) = \mathcal{G}(\alpha a)$$

and applying the corollary of Lemma 15, we obtain:

$$\frac{\eta(\lambda_* a, \mathcal{G}(\gamma 0))}{\eta(\lambda_* a, \mathcal{C}(\gamma 0))} = \frac{\eta(\lambda_* a, \mathcal{G}(\gamma 0))}{\eta(\lambda_* a, \Pi^{\eta}_{b,a}(\mathcal{C}(\gamma' f 0)))} = r$$

$\lambda : [a; b] \to \mathbb{R}^4$ being a lightlike geodesic of η between $\gamma 0$ and $\gamma' f 0$. Without loosing generality we can write:

$$\lambda_* a = \left(\frac{\partial}{\partial x^0} + a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + a^3 \frac{\partial}{\partial x^3} \right)(\lambda a), \text{ with } {a^1}^2 + {a^2}^2 + {a^3}^2 = 1$$

(lightlike). It follows that:

$$r = \delta(1 - \beta a^3).$$

Transforming to spherical coordinates $(x^0, x^1, x^2, x^3) \to (x^0, x^r, x^\vartheta, x^\varphi)$ (see Appendix) it can be seen that $c\beta_r \equiv -\beta c a^3$ represents the radial component of the relative velocity between emitter and receiver.

Now $r = \delta(1 + \beta\gamma)$, which is a well-known result (see [10], 2 - 2).

For observers moving away from each other ($\beta_r > 0$) a redshift will be observed ($r > 1$). In a similar way, a relative approximation implies an observed blue--shift.

Example 6. Let Q be a FRF on a time-oriented CST $(M, \mathscr{C})$, with $\Phi : T \times S \to U \subset M$. Remember that $\Phi_* \partial_t = Q$, $\Phi^* g^Q = -\mathrm{d}t^2 \oplus fh$, for some function $f > 0$ defined on $T \subset R$. Our aim now is to compute the proper time ratio r between two observers of Q. (We do the computation in $(T \times S, [-\mathrm{d}t^2 \oplus fh])$).

Defining $t' := \int_0^t f^{-1/2} \mathrm{d}t$, we obtain $\partial_{t'} = f^{1/2} \partial_t$ and $\mathrm{d}t' = f^{-1/2} \mathrm{d}t$. From the expression: $-1 = g^{\partial t'}(\partial_{t'}, \partial_{t'}) = f g^{\partial t'}(\partial_t, \partial_t)$ we deduce that $g^{\partial t'} = f^{-1} g^{\partial t} = -f^{-1} \mathrm{d}t^2 \oplus h = -\mathrm{d}t'^2 \oplus h$; then $\partial_{t'}$ is a temporally homogeneous FRF. Because of theorem 1, the proper time ratio r' between observers of $\partial/\partial t'$ is equal to 1.

But, if we consider $g \in [-dt^2 \oplus fh]$ and λ lightlike geodesic of g, we get (from Lemma 15)

$$r = \frac{g(\lambda_* a, \partial_t \lambda a)}{g(\lambda_* b, \partial_t \lambda b)} = \frac{(f \circ t(\lambda a))^{-1/2} g(\lambda_* a, \partial_{t'}(\lambda a))}{(f \circ t(\lambda b))^{-1/2} g(\lambda_* b, \partial_{t'}(\lambda b))} =$$

$$= \left(\frac{f \circ t(\lambda b)}{f \circ t(\lambda a)}\right)^{1/2} r' = \left(\frac{f \circ t(\lambda b)}{f \circ t(\lambda a)}\right)^{1/2}.$$

This result is more general than the one obtained for the cosmological red shift in the Robertson-Walker models (see [10]), because (spatial) homogeneity and isotropy are not imposed to the h. In such models the factor $\sqrt{f}$ is known as the «expansion factor». A value $r > 1$ means, because the metric in each of the spatial hypersurfaces is given by $f(t)h$, that «the universe is expanding».

5. THE SEGAL'S MODEL

In this section we are going to describe briefly the cosmological model of I.E. Segal. The reader will find in [2] a detailed discussion of the physical and observational features on which the model is based, and the proofs of some of its fundamental results. We have considered convenient to change the terminology used by Segal (sometimes obscure from the point of view of differential geometry) into the notations developed in sections 2 and 3 of this paper. We shall begin with the basic assumptions of the Segal's model ([2], ch. III - 1, 2), which consists in a *triplet* $(M, \mathscr{C}, Q)$. *$(M, \mathscr{C})$ being a time-oriented CST* (3) *verifying the causality condition and Q being a certain global complete FRF on $(M, \mathscr{C})$, spatially homogeneous and isotropic and temporally homogeneous.*

The hypothesis about the spatial homogeneity and isotropy of the FRF Q, in addition to be a reasonable requirement to approach in a simplified way many of the analysis of cosmological observational data, is also a consequence of applying the «cormological principle» to the observations done on Earth, around

(3) Initially Segal postulates what he calls an «infinitesimal causal orientation» for the spacetime. Later he takes the «cones» of that structure as being induced by locally Lorentzian metrics (or, equivalently, by a time-oriented conformal structure). The fact that Segal's discussion about the impossibility of inducing those «cones» by Finslerian non-Lorentzian metrics is not conclusive for dimension 4 has pushed us to include here explicitly the *time-oriented conformal structure* among the *basic* assumptions of the model.

which the universe appears, on a large scale, very aproximately isotropic (see the comment to Lemma 11). In this sense, Q is that FRF whose integral curves represent the mean trajectories of matter in spacetime. Temporal homogeneity is necessary for the flux of the mentioned FRF to be a one-parameter subgroup of the causal group of spacetime (see Lemma 11b). In [2] physical arguments are given which, after Segal, favour the latter postulate: the obtention of a time-independent «state» concept, the notion of a well defined «energy», and an energy conservation law. Note that classical cosmological models do not satisfy this property, because including temporal homogeneity for «comoving» reference frames imply, in General Relativity, that the metric spacetime is static and cannot offer a convincing explanation of the observed cosmological red shift (see example 6).

It can be proved that there exist only three spacetimes which satisfy these assumptions ([2], p. 58 - 59), namely:

1) $(\mathbb{R}^4, [\eta])$ (see examples 1 and 3), conformal version of Minkowski space, having a causal group of dimension 11 [8].

2) A CST causally diffeomorphic to a certain open submanifold of $(S^1 \times S^3, [-l \oplus h])$ (see example 2), having a causal group of dimension 10.

3) $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ (examples 2 and 4) which has a causal group of dimension 15 (see Lemma 21 below), i.e., maximal.

Segal argues that, as there exist causal diffeomorphisms of the first two (Lemmas 18 and 19 below) into open subsets of causal manifolds which are covered by the third one, and as the causal groups of the first two are essentially subgroups of the causal group of the latter one, $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ *must be considered «universal», and thus it must be taken as a spacetime for a cosmological model.*

Concerning the FRF postulated in the model, as we said in example 4 it must be conjugate to the field ∂_t on $\mathbb{R} \times S^3$; so we do not loose generality (see Lemma 14) by taking ∂_t as the FRF of the model.

Now we are going to give the definitions and basic results which allow to consider the Segal's CST $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ as the universal covering (4) of the conformal compactification of Minkowski space.

DEFINITION 12.

(a) Let b be a scalar product on $\mathbb{R}^6$ with signature $(--++++)$. Let us consider the projective space $P\mathbb{R}^6$; we shall denote by $\tilde{\omega} \in P\mathbb{R}^6$, with $\omega \neq 0 \in$

(4) When we say that a time-oriented CST «covers» another one it is understood that the covering map preserves the conformal structure and time orientability (see Lemma 4).

$\in \mathbb{R}^6$, the class $\tilde{\omega} := \{y \in \mathbb{R}^6 / y = \lambda\omega; \lambda(\neq 0) \in \mathbb{R}\}$. We define *the projective quadric B induced by b* as the set;

$$B := \{\tilde{\omega} \in P\mathbb{R}^6 / b(\omega, \omega) = 0\}.$$

(b) Let us consider the group $GL(6, \mathbb{R})$; each element $T \in GL(6, \mathbb{R})$ induces a transformation $\tilde{T}: P\mathbb{R}^6 \to P\mathbb{R}^6$, $\tilde{\omega} \mapsto (\widetilde{T\omega})$. Let be the subgroup of $GL(6, \mathbb{R})$

$$0(2,4) := \{T \in GL(6, \mathbb{R}) / b(Tw, Ty) = b(w, y),\ \forall w, y \in \mathbb{R}^6\}.$$

We define the *projective group* $0(B)$ *of the quadric B* as the set

$$0(B) := \{\tilde{T} / T \in 0(2,4)\}.$$

(It is clear that $0(B)$, acting on the quadric B, keeps B invariant).

LEMMA 16. *Let B and* $0(B)$ *be as in definition 12. Then we have:*

(a) *B has the structure of a four dimensional* C^∞ *compact manifold.*

(b) $0(B)$ *has the structure of a Lie group, isomorphic to* $0(2,4)/\mathbb{Z}_2$, *and under whose action B is an homogeneous space.*

(c) *There exists an unique* conformal structure $\mathscr{C}_B$ *on B such that the action of* $0(B)$ *on B is conformal (i.e.:* $\varphi^* \mathscr{C}_B = \mathscr{C}_B$, $\forall \varphi \in 0(B)$).

Proof. See [2], scholium 2.9, page 38. ■

LEMMA 17. *Let* $(S^1 \times S^3, [-l \oplus h])$ *be the time-oriented CST from example 2. Let* $(B, \mathscr{C}_B)$ *be a CST as above. Then it holds:*

(a) *The map* $\Pi : S^1 \times S^3 \to B$, $((u^{-1}, u^0), (u^1, u^2, u^3, u^4)) \mapsto \tilde{\omega}$, *with* $(u^{-1})^2 + (u^0)^2 = 1 = (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2$ *and* $\omega = (u^{-1}, u^0, u^1, u^2, u^3, u^4) \in \mathbb{R}^6$, *is a double covering and it is conformal (i.e.:* $\Pi^* \mathscr{C}_B = [-l \oplus h]|_{\Pi^{-1}(B)}$).

In fact $(B, \mathscr{C}_B)$ *is conformally diffeomorphic to* $(S^1 \times S^3 / \{I, -I\}, [-l \oplus h])$, *I (resp.* $-I$) *being the identity (resp. the antipodal map) on* $S^1 \times S^3$.

(b) *The time-orientation in* $(S^1 \times S^3, [-l \oplus h])$ *(example 2) is invariant under the antipodal map* $-I$; *thus* the CST $(B, \mathscr{C}_B)$ becomes time-oriented *and the mapping* Π *is causal.*

(c) *The causal group of* $(B, \mathscr{C}_B)$ *is isomorphic to* $SO_0(2,4)/\mathbb{Z}_2$ *(thus maximal, see Lemma 3), under which B is still an homogeneous space.*

Proof. See [2], scholia 2.10 and 2.11. p. 39 - 40. ■

LEMMA 18. *Let be the time-oriented CST* $(B, \mathscr{C}_B)$ *as above and let* $(\mathbb{R}^4, [\eta])$ *be like in example 1. Then the map:*

$$j:(\mathbb{R}^4,[\eta])\longrightarrow(B,\mathscr{C}_B),\ x\longmapsto\tilde{\omega},\qquad \textit{with}$$

$$\omega=\left(1+\frac{\eta(x,x)}{4},x^0,x^1,x^2,x^3,1-\frac{\eta(x,x)}{4}\right)\in\mathbb{R}^6,$$

is a causal diffeomorphism onto its image, which is dense in B.

Proof. See [2], scholium 2.12 and corollary 2.13.1, pages 42 - 43. ∎

We shall call $(B,\mathscr{C}_B)$ the *conformal compactification of the Minkowski space.*

LEMMA 19. $(\mathbb{R}^1\times S^3,[-\mathrm{d}t^2\oplus h])$, *as in example 2, is the universal covering of the conformal compactification of the Minkowski space.*

Proof. It follows from Lemmas 16 and 17 and from the composition $\Pi\circ\Psi$ of causal maps

$$(\mathbb{R}\times S^3,[-\mathrm{d}t^2\oplus h])\xrightarrow{\Psi}(S^1\times S^3,[-l\oplus h])\xrightarrow{\Pi}(B,\mathscr{C}_B)$$

(Ψ is defined in example 2 and Π in Lemma 17, (a)). ∎

REMARK. If we denote by s the section of the covering map $\Pi\circ\Psi$ which takes the point:

$$\tilde{\omega}\in B\quad(\text{with}\quad\omega=(1,0,0,0,0,1)\in\mathbb{R}^6)$$

into the point:

$$(t=0,\quad u^1=0,\quad u^2=0,\quad u^3=0,\quad u^4=1)\in\mathbb{R}\times S^3,$$

then s is given by

$$\begin{cases} t=\arcsin\dfrac{\omega^0}{({\omega^{-1}}^2+{\omega^0}^2)^{1/2}} \\ \\ u^i=\dfrac{\omega^i}{({\omega^{-1}}^2+{\omega^0}^2)^{1/2}}\,. \end{cases}\qquad (i=1,2,3)$$

We can summarize the maps we have introduced up to now in the following diagram

$$
\begin{array}{ccccc}
\mathbb{R}^6 \longrightarrow P\mathbb{R}^6 \supset & (B, \mathscr{C}_B) & \xleftarrow{\;j\;} & (\mathbb{R}^4, [\eta]) \\
 & \uparrow \Pi & & \\
s & (S^1 \times S^3, [-l \oplus h]) & & \\
 & \uparrow \Psi & & \\
 & (\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h]). & &
\end{array}
$$

Thus the composition $s \circ j : (\mathbb{R}^4, [\eta]) \to (\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ is a causal diffeomorphism onto its image (see the foregoing Lemmas 18 and 19).

We are going to prove also that $s \circ j$ is the well known conformal map of the Minkowski space into a region of the static Einstein space $(\mathbb{R} \times S^3, -\mathrm{d}t^2 \oplus h)$, given for example in [6], *p. 118 - 121.*

LEMMA 20.

(i) *The map $(s \circ j) : \mathbb{R}^4 \to \mathbb{R} \times S^3$ is given, in the corresponding charts (see Appendix) by the expressions:*

$$
\begin{cases}
t = \arctan \dfrac{x^0}{1 + \dfrac{\eta(x,x)}{4}} \\[2ex]
\rho = \arctan \dfrac{x^r}{1 - \dfrac{\eta(x,x)}{4}}
\end{cases}
\qquad
\begin{cases}
\vartheta = x^\vartheta \\
\varphi = x^\varphi
\end{cases}
$$

and it establishes a diffeomorphism between $\mathbb{R}^4$ and the open set $U \subset \mathbb{R} \times S^3$ defined by

$$
\begin{cases}
-\pi < t + \rho < \pi \\
-\pi < t - \rho < \pi \\
0 \leqslant \vartheta \leqslant \pi \\
0 \leqslant \varphi \leqslant 2\pi
\end{cases}
\qquad \textit{with} \quad \rho \geqslant 0
$$

the inverse diffeomorphism being given by:

$$
\begin{cases}
x^0 = \tan\left(\dfrac{t+\rho}{2}\right) + \tan\left(\dfrac{t-\rho}{2}\right) = \dfrac{2\sin t}{\cos t + \cos\rho} \\[2ex]
x^r = \tan\left(\dfrac{t+\rho}{2}\right) - \tan\left(\dfrac{t-\rho}{2}\right) = \dfrac{2\sin\rho}{\cos t + \cos\varphi}
\end{cases}
\qquad
\begin{cases}
x^\vartheta = \vartheta \\
x^\varphi = \varphi
\end{cases}
$$

(ii) *It holds*

$$(s \circ j)^{-1} {}^* \eta = \frac{1}{\cos^2\left(\dfrac{t+\rho}{2}\right)\cos^2\left(\dfrac{t-\rho}{2}\right)} (-\mathrm{d}t^2 \oplus h)|_U .$$

Proof. The composition $(s \circ j)$ is given, in the usual coordinates (x^0, x^1, x^2, x^3) of $\mathbb{R}^4$, by:

$$(x^0, x^1, x^2, x^3) \xrightarrow[\text{Lemma 18}]{j} \tilde{\omega} \longrightarrow$$

$$\text{with: } \omega := \left(1 + \frac{\eta(x, x)}{4}, x^0, x^1, x^2, x^3, 1 - \frac{\eta(x, x)}{4}\right)$$

$$\xmapsto[\text{(Lemma 17)}]{\text{section of } \Pi} k_x\left(1 + \frac{\eta(x, x)}{4}, x^0, x^1, x^2, x^3, 1 - \frac{\eta(x, x)}{4}\right) \longmapsto$$

$$\text{with } k_x := \left(\left(1 + \frac{\eta(x, x)}{4}\right)^2 + x^{0^2}\right)^{-1/2}$$

$$\xmapsto[\text{(Ex. 2)}]{\text{section of } \Psi} \left(t, u^1 = k_x x^1, u^2 = k_x x^2, u^3 = k_x x^3, u^4 = k_x\left(1 - \frac{\eta(x, x)}{2}\right)\right)$$

$$\text{with } t = \operatorname{arc\,tan} \frac{x^0}{1 + \dfrac{\eta(x, x)}{4}} .$$

Taking into account the changes (see Appendix) $(x^0, x^1, x^2, x^3) \mapsto (x^0, x^r, x^\vartheta, x^\varphi)$ and $(t, u^1, u^2, u^3) \mapsto (t, \rho, \vartheta, \varphi)$, the desired expression for $(s \circ j)$ follows. The rest is trivial by using certain trigonometrical identities. ■

We next summarize the basic result about the causal group of the Segal's CST.

LEMMA 21. *The causal group of the time-oriented CST* $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ *is isomorphic (as Lie group) to the group* $\widetilde{SU}(2, 2)$ *(*$\tilde{G}$ *denotes here the universal covering of* G*).*

Proof. It follows from the following facts:

(i) $(B, \mathscr{C}_B)$ is a time-oriented CST (Lemma 17.b), on which the connected group $SO_0(2, 4)/\mathbb{Z}_2$ acts causally and transitively (Lemma 17.c).

(ii) $SO_0(2, 4)/\mathbb{Z}_2$ is isomorphic to $SU(2, 2)/\mathbb{Z}_4$.

(iii) The universal covering map

$$\Pi \circ \Psi : (\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h]) \to (B, \mathscr{C}_B)$$

is causal (example 2 and Lemma 17.b).

(iv) [2], scholium 2.5. page 33. ■

The causal group of the Segal's CST is then maximal (see Lemma 3). The expressions for the 15 fundamental fields generated by the action of the causal group are given in the Appendix (5).

Now the question is the following: given an observer γ of ∂_t, *which are the coordinates γ assigns to its neighboring points?*. That question is important. In fact, the analysis of the experimental observations being essentially Minkowskian (i.e., in terms of $(\mathbb{R}^4, [\eta])$), the problem is how to «transfer» the interesting events from $(M, \mathscr{C})$ to the conformal vector space $(T_pM, \mathscr{C}_p)$ (canonically identified with $(\mathbb{R}^4, [\eta])$, for every point p on the range of γ).

Let us remember briefly how the local coordinate system associated to an observer is built in general relativity. In this theory, space-time is represented by a Lorentzian manifold (M, g). Given an observer $\gamma : \mathbb{R} \supset \epsilon \to M$ (future-directed timelike curve such that $g(\gamma_*, \gamma_*) = -1$, not necessarily geodesic) the procedure goes as follows ([9], 13.6; [1], 2.10):

(i) A point $c \in \epsilon$ is chosen as origin of proper times (on the curve γ proper time is given by arc lenght).

(ii) Given $a \in \epsilon$, the bunch of geodesics orthogonal to $\gamma_*(a)$ at γa generate (locally) a spacelike hypersurface S_a. The validity of the construction requires that the «neighbouring» region of $\gamma\epsilon$ to be coordinatized can be foliated by a family of such hypersurfaces.

(iii) Four C^∞ ortonormal vector fields on γ, e_μ ($\mu = 0, 1, 2, 3$), are chosen with $e_0 = \gamma_*$.

(iv) Each event A in the vicinity of $\gamma\epsilon$ can be located by means of four coordinates:

(5) It is possible to give an alternative description of the Segal's model in terms of Lie groups [2]. In that description one begins by defining time-oriented conformal structures on both the space $H(2)$ of (2×2) complex hermitian matrices (considered as a manifold) and the Lie group $\tilde{U}(2)$ (universal covering of $U(2)$). It can be shown that both CST'S are causally diffeomorphic to $(\mathbb{R}^4, [\eta])$ and $(\mathbb{R} \times S^3, [-\mathrm{d}t^2 \oplus h])$ respectively. The analogue rôle to the above-mentioned map j is played here by the so-called Cayley map, which transforms hermitian matrices into unitary ones. Such a description, although interesting, is not necessary at all to study the Segal's model.

$x^0 = a - c$ (S_a being the spacelike hypersurface through A).

x^i $(i = 1, 2, 3) = i^{th}$ «spacelike» coordinate of A given by the normal chart (defined around γa by the exponential map of the Levi-Civita connection ∇^g) associated to the basis $e_\mu(a)$ of $T_{\gamma a}M$.

Going back to the Segal's cosmological model ($\mathbb{R} \times S^3$, $[-dt^2 \oplus h]$, ∂_t), let us consider the induced Lorentzian metric $g^{\partial_t} \in [-dt^2 \oplus h]$ (Lemma 8) which is nothing else but $-dt^2 \oplus h$. Given an observer $\gamma : \mathbb{R} \to M$ of ∂_t, it is straightforward to see (as a consequence of being ∂_t a temporally homogeneous FRF) that the local coordinates associated to γ in the above mentioned sense *coincide* (up to traslations in x^0, and provided that the choice of the «spacelike» fields e_i $(i = 1, 2, 3)$ is a suitable one) with the ones assigned by the normal chart (defined around any point γa by the exponential map of $-dt^2 \oplus h$) associated to the basis $e_\mu(a)$ of $T_{\gamma a}M$ (this normal chart takes the form (Id, x), x being a normal chart of h on S^3). *On the metric manifold* ($\mathbb{R} \times S^3, g^{\partial_t}$), *which corresponds to the static spacetime of Einstein, the mapping* $\mathrm{Exp}^{-dt^2 \oplus h}$ (or, more precisely, the inverse of this exponential in the region where it is a diffeomorphism) *appears as the key-map giving the local coordinates assigned by the observers of* ∂_t. Let $V_1 \subset \mathbb{R}^4$ and $V \subset \mathbb{R} \times S^3$ be open sets defined, in the corresponding charts on $\mathbb{R}^4$ and $\mathbb{R} \times S^3$ (see Appendix), respectively by

$$\begin{cases} -\infty < x^0 < \infty \\ 0 \leqslant x^r < \pi \\ 0 \leqslant x^\vartheta \leqslant \pi \\ 0 \leqslant x^\varphi \leqslant 2\pi \end{cases} \quad \text{and} \quad \begin{cases} -\infty < t < \infty \\ 0 \leqslant \rho < \pi \\ 0 \leqslant \vartheta \leqslant \pi \\ 0 \leqslant \varphi \leqslant 2\pi. \end{cases}$$

It is very easy to see that the above exponential map defines a diffeomorphism $V_1 \to V$, given by

$$\begin{cases} x^0 = t \\ x^r = \rho \end{cases} \qquad \begin{cases} x^\vartheta = \vartheta \\ x^\varphi = \varphi \end{cases}$$

whose inverse is the coordinate defining map.

But in the Segal's cosmological model there is no metric having an intrinsic meaning on the space-time. How to define then the local coordinate system associated to the observers of ∂_t? *Segal's criterion is that this coordinatization is given by the* map $s \circ j : \mathbb{R}^4 \to \mathbb{R} \times S^3$ (more precisely, by the inverse of this map in the region where it is a diffeomorphism); this mapping has been described in Lemma 20.

Concerning this alternative coordinatization the following remarks should be made:

(i) In the Segal's model, it is in principle a matter of taste to postulate a coordinatization by means of $\mathrm{Exp}^{-dt^2 \oplus h}$ or to do it by means of $(s \circ j)$.

(ii) The map $(s \circ j)$ is not only a diffeomorphism of the *whole* $\mathbb{R}^4$ onto its image $\mathscr{U}$ (on the contrary, $\mathrm{Exp}^{-dt^2 \oplus h}$ only provides a diffeomorphism of a certain neighborhood V_1 of the origin of $\mathbb{R}^4$ onto its image V) but it is also *globally causal* (Lemmas 18 and 19) (on the contrary, $\mathrm{Exp}^{-dt^2 \oplus h}$ is only *locally causal*, see [2], scholium 2.3). From that point of view, the second coordinatization seems then to have «better» properties.

(iii) From the coordinate expressions of $\mathrm{Exp}^{-dt^2 \oplus h}$ and $(s \circ j)$ (see above) it is easily deduced that, in the vicinity of the point of observation, both coordinatizations lead to the same results (up to terms of second order in the coordinates). Therefore, Segal argues, one must look for *experimental observations of events taking place in remote regions* in order decide about the *physical* validity of each one of these coordinatizations.

(iv) Segal claims that his model, with the second coordinatization,

(a) *predicts a cosmological redshift.*

(b) Fits the experimental data of cosmological redshift *better* than the classical (metric) «expansive» models do.

Without going into questions related to (b) (see [11]), *we shall see in section 6 that, actually, the coordinatization by means of* $(s \circ j)$ *does not predict any cosmological redshift at all.*

But, before analyzing the claimed cosmological reshift in the Segal's model we are going to see how the coordinatization via $(s \circ j)$ can be expressed in the language of the reference frames developed in section 3.

LEMMA 22. *Let* $(\mathbb{R} \times S^3, [-dt^2 \oplus h])$ *be the time-oriented CST of Segal's model. Let* $(s \circ j) : \mathbb{R}^4 \to \mathscr{U} \subset \mathbb{R} \times S^3$ *be the above mentioned diffeomorphism (Lemma 20). Let us define on* $\mathscr{U}$ *the vector field* $\mathcal{Q} := (s \circ j)_* \partial_{x^0}$*; then it holds.*

(i) *The Lorentzian metric* $g^{\mathcal{Q}}$ *induced on* $\mathscr{U}$ *(Lemma 8) is* $g^{\mathcal{Q}} = F(-dt^2 \oplus$ $\oplus\, h)|_{\mathscr{U}}$ *with* $F := \left(\cos^2 \dfrac{t+\rho}{2} \quad \cos^2\left(\dfrac{t-\rho}{2}\right)\right)^{-1}$

(ii) $\mathcal{Q}$ *is a temporally homogeneous FRF.*

(iii) *The exponential* $\mathrm{Exp}^{F(-dt^2 \oplus h)}$ *associated to* $g^{\mathcal{Q}}$ *in a neighborhood of the point* P *of* $\mathbb{R} \times S^3$ *with coordinates* $(t = 0,\ \rho = 0,\ \vartheta = 0,\ \varphi = 0)$ *coincides with the map* $(s \circ j)$.

(iv) *The vector field* $\mathcal{Q}$ *satisfies:*

$$\mathcal{Q} = \left(\frac{1}{2}(1 + \cos t \cos \rho)\,\partial_t - \frac{1}{2} \sin t \sin \rho\ \partial_\rho\right)\Big|_{\mathscr{U}}.$$

Proof.

(i) Because $(s \circ j)^{-1*}\eta = F(-\mathrm{d}t^2 \oplus h)$ (Lemma 20.ii), it holds:

$$F(-\mathrm{d}t^2 \oplus h)(\mathcal{Q}, \mathcal{Q}) = \eta((s \circ j)^{-1}_{*}\mathcal{Q}, (s \circ j)^{-1}_{*}\mathcal{Q}) = \eta(\partial_{x^0}, \partial_{x^0}) = -1.$$

(ii) This is a consequence of Lemma 14. The proper time function is

$$x^0 \circ (s \circ j)^{-1} = \frac{2 \sin t}{\cos t + \cos \rho}.$$

Moreover $\mathcal{Q}$ is complete.

(iii) Follows trivially from the fact that $(s \circ j) : (\mathbb{R}^4, \eta) \to (\mathcal{U}, F(-\mathrm{d}t^2 \oplus h)|_{\mathcal{U}})$ is an isometry.

(iv) Writing $\mathcal{Q} = (f\,\partial_t + g\,\partial_\rho)|_{\mathcal{U}}$, and into account

$$\begin{cases} \dfrac{\partial x^0}{\partial t} = 2\,\dfrac{1 + \cos t \cos \rho}{(\cos t + \cos \rho)^2} = \dfrac{\partial x^r}{\partial \rho} \\[2ex] \dfrac{\partial x^r}{\partial t} = 2\,\dfrac{\sin t \sin \rho}{(\cos t + \cos \rho)^2} = \dfrac{\partial x^0}{\partial \rho} \end{cases}$$

and the identity $(1 + \cos t \cos \rho)^2 - (\sin t \sin \rho)^2 = (\cos t + \cos \rho)^2$, the desired expression follows. ∎

The foregoing Lemma offers a new perspective on the Segal's cosmological model. In fact, the observers of $Q = \partial_t$ give coordinated (locally) to the spacetime manifold $\mathbb{R} \times S^3$ essentially in the same way as it is done in general relativity, i.e., *via the exponential map associated to a certain Lorentzian metric* compatible with the conformal structure $[-\mathrm{d}t^2 \oplus h]$. The new features of the model are not in the way of «giving coordinates», but in the choice of the metric; Segal postulates that «anthropomorphically possible local measurements are represented theoretically by *flat* rather than *curved* variables; while on the other hand, the *true* nonanthropomorphic dynamics and analysis are curved» ([2], page 75; here the adjective «flat» refers obviously to the fact that $F(-\mathrm{d}t^2 \oplus h)$ is a flat Lorentzian metric, while $(-\mathrm{d}t^2 \oplus h)$ is not), what is equivalent to say that *each observer of $Q = \partial_t$ places himself «as if» the FRF of which he is an integral curve would be actually the corresponding $\mathcal{Q}$ centered at the point of observation and not Q.*

The following diagram summarizes all we have seen about the FRF's Q and $\mathcal{Q}$:

$$\mathbb{R}\times S^3 \xrightarrow{\Phi = \mathrm{Id}} \mathbb{R}\times S^3 \supset \mathcal{V} \xleftarrow{\mathrm{Exp}_p^{-dt^2\oplus h}} \mathcal{V}_1 \subset \mathbb{R}^4$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$Q = \partial_t \qquad\qquad \partial_{x^0}$$

$$g^Q = -\mathrm{d}t^2 \oplus h \qquad\qquad g^{\partial_{x^0}} = -\mathrm{d}x^{0^2} \oplus \sum_i (\mathrm{d}x^i)^2 \equiv \eta$$

$$\mathbb{R}^4 \simeq \mathbb{R}^1\times\mathbb{R}^3 \xrightarrow[(s\circ j)]{\tilde{\Phi} = \mathrm{Exp}_p^{F(-dt^2\oplus h)}} \mathcal{U}\subset \mathbb{R}\times S^3$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\partial_{x^0} \qquad\qquad \mathbb{Q} = (s\circ j)_* \partial_{x^0}$$

$$g^{\partial_{x^0}} = -\mathrm{d}x^{0^2} \oplus \sum_i (\mathrm{d}x^i)^2 \equiv \eta \qquad g^{\mathbb{Q}} = F(-\partial t^2 \oplus h)$$

Note

The maps Φ and $\tilde{\Phi}$ are the diffeomorphisms making the fields Q and $\mathbb{Q}$ temporally homogeneous FRF (Def. 9). The exponentials $\mathrm{Exp}_p^{g^Q}$, $\mathrm{Exp}_p^{g^{\mathbb{Q}}}$ (the last one coincides with $\tilde{\Phi}$) induce the respective coordinatizations.

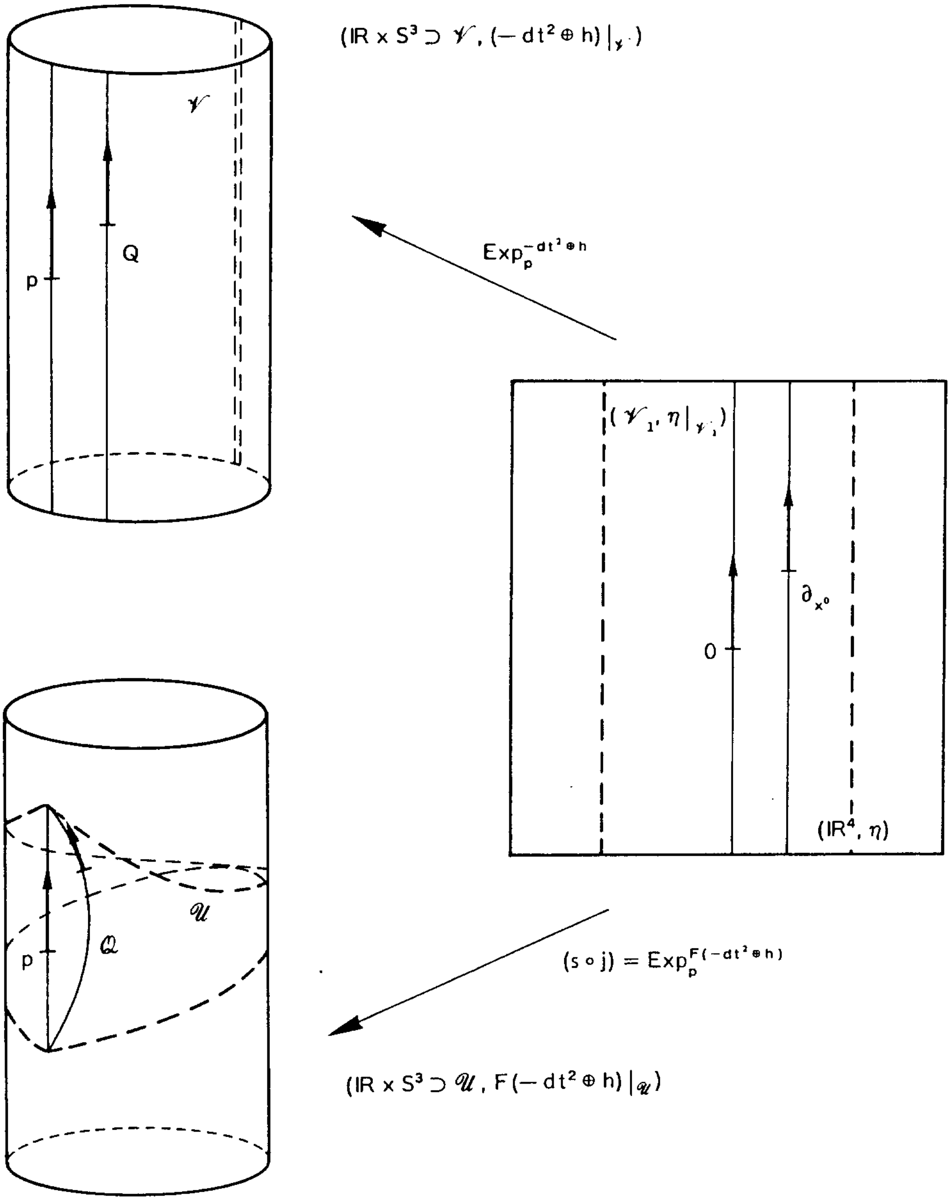
Finally the drawing in the next page both ways of transfering the vector field ∂_{x^0} from Minkowski specetime (where the analysis of the experimental observations takes place) to Segal's spacetime $(\mathbb{R}\times S^3, [-\mathrm{d}t^2\oplus h])$.

6. DOES THE SEGAL'S MODEL PREDICT A COSMOLOGICAL REDSHIFT?

As we have seen in section 4, a time-oriented CST $(M, \mathscr{C})$ with a global FRF Q whose integral curves represent the mean trajectories of matter provides an adequate framework to analyze the cosmological redshift (i.e., redshift between observers of Q). The Segal's model follows this scheme; and in that sense *the following theorem is the most important result of this work.*

THEOREM 2. *Let* $(\mathbb{R}\times S^3, [-\mathrm{d}t^2\oplus h], \partial_t)$ *be the Segal's cosmological model. Let* γ *and* γ' *be two observers of* ∂_t. *Then the proper time ratio for the radiation emitted by* γ *and received by* γ' *is equal to 1.*

Proof. It follows immediately from the temporal homogeneity of the FRF ∂_t and from the theorem 1. ■


$(\mathbb{R} \times S^3 \supset \mathscr{V}, (-dt^2 \oplus h)|_{\mathscr{V}})$
$\mathscr{V}$
Q
p
$\mathrm{Exp}_p^{-dt^2 \oplus h}$
$(\mathscr{V}_1, \eta|_{\mathscr{V}_1})$
∂_{x^0}
0
$(\mathbb{R}^4, \eta)$
$\mathscr{U}$
$\mathscr{Q}$
p
$(s \circ j) = \mathrm{Exp}_p^{F(-dt^2 \oplus h)}$
$(\mathbb{R} \times S^3 \supset \mathscr{U}, F(-dt^2 \oplus h)|_{\mathscr{U}})$

REMARK. Given any $t_0 \in \mathbb{R}$, the non-uniqueness of the path followed by light between γt_0 and the observer γ' does not change at all that result.

The foregoing theorem establishes that, in the Segal's model there is no cosmological redshift.

This contradicts Segal's claim, and we are going now to analyze in detail the arguments given by Segal to support the existence of redshift in his model; and we shall follow three different approaches to that question.

Previous remark

(i) From now on we shall identify the observers γ and γ' of $Q \equiv \partial_t$ with a typical galaxy and with the Earth respectively (see section 4).

(ii) We shall denote by G some point on the range of γ (corresponding to a certain proper time τ_0, $\gamma\tau_0 \equiv G$). Without loosing generality we can take $t(G) = \rho(G) = \vartheta(G) = \varphi(G) = 0$. We shall assume that, on a neighborhood of G, the galaxy emits radiation which is received at Earth on a neighborhood of some point on the range of γ' which will be denote by T. We shall call λ the light path between G and T with the parametrization making it (see Section 4) a geodesic of the global Lorentzian metric g^{∂_t}. Without loosing generality we shall consider that this propagation takes place with $\vartheta = \varphi = \text{constant} = 0$ (i.e., with $u^1 = u^2 = \text{constant} = 0$), so that we can consider λ as an integral curve of the (g^{∂_t}-geodesic) field $\partial_t + \partial_\rho$. Choosing $\lambda 0 = G, \lambda\alpha = T$, the coordinates of the point T will be $t(T) = \rho(T) = \alpha\, (0 < \alpha < \pi)$, $\vartheta(T) = \varphi(T) = 0$.

Finally, we shall denote by P the point on γ with coordinates $t(P) = \alpha\ (0 < \alpha < \pi)$, $\rho(P) = \vartheta(P) = \varphi(P) = 0$.

(iii) We shall call *Segal-chart centered at* G the map $(s \circ j)_G^{-1} : \mathbb{R} \times S^3 \supset \mathcal{U}_G \to \mathbb{R}^4$ discussed in section 5. The subcript G means that the point G is taken as the origin of the coordinate system. We shall denote by $(x_G^0, x_G^1, x_G^2, x_G^3)$ the cartesian coordinates in the $\mathbb{R}^4$ of the Segal-chart centered at G and by $\mathcal{Q}_G$ the vector field (with domain $\mathcal{U}_G$) defined by this chart (see Lemma 22).

Whenever we refer to the Segal-chart centered at P we shall use the notation $(s \circ j)_P^{-1}, (x_P^0, x_P^1, x_P^2, x_P^3), \mathcal{U}_P, \mathcal{Q}_P$; and similarly for the point T.

The point of view supporting the existence of a cosmological redshift in the Segal's model can be summarized in the following way. If the observer γ' gives coordinates to the manifold by means of the exponential map associated to $g^Q = -\mathrm{d}t^2 \oplus h$, the integral curves of Q (which represent the mean trajectories of matter) will correspond (on the $\mathbb{R}^4$ of the chart) to the integral curves of the vector field ∂_{x^0}: *the cosmos appears to be «static» to the observer* γ'. But (Segal's postulate) γ' assigns coordinates through the map $(s \circ j)$, which implies that the integral curves of Q correspond (on the $\mathbb{R}^4$ of the chart) to the ones of the vector field $\left(1 + \frac{\eta(x, x)}{4}\right)\partial_{x^0} + \frac{x^0}{2} S$ (see Appendix): *the cosmos does not appear to be static to the observer* γ' (see later, paragraph B). So, this alternative way to give coordinates (the only «anthropomorphically possible», according to Segal) would be responsible for the existence of a cosmological redshift.

We are now going to develop three kind of arguments which show that the above point of view seems to be incorrect and which confirm the result obtained in theorem 2.

A) The first argument is structural-like. The latter point of view *can not be right*, simply because different ways (all of them more or less «subjective», although perhaps only one «antropomorphically possible») of coordinatizing the manifold can not change the result («objective», as related to the notions of *proper time* on γ and γ') given by theorem 2 for the proper time ratio r.

REMARK. To argue that, instead of giving coordinates to the manifold via $(s \circ j)$, the observer on Earth «observes the mean matter as if it were following the integral curves of the vector field $\mathcal{Q} := (s \circ j)_* \partial_{x^0}$» *has simply no sense at all*. In fact, *either* the trajectories of the mean matter are integral curves of the global FRF $Q = \partial_t$ (that is implied straightforwardly from the postulates of the Segal's model), *or* they are integral curves of the local FRF $\mathcal{Q}$ (both possibilities being mutually excluding). Anyway, even assuming that we don't require the global FRF Q to be the one whose integral curves represent the average trajectories of matter in spacetime, it must be remarked that also $\mathcal{Q}$ is *temporally homogeneous* (see Lemma 22), so that the conclusion of theorem 2 is still valid for any two observers of $\mathcal{Q}$ (actually, any vector field $\mathscr{A}_* \partial_{x^0}$, with $\mathscr{A} : (\mathbb{R}^4, [\eta]) \to (\mathbb{R} \times S^3, [-dt^2 \oplus h])$ some causal diffeomorphism onto its image, is a temporally homogeneous FRF; see Lemma 14).

B) The second argument is related to the *observational appearance* of γ with respect to γ'. The question is to analyze *in what sense the cosmos does not appear to be static to the observer* γ'. In order to do that we shall compute the jacobian matrix at G of the change between Segal-charts centered at G and T (assuming that each one of these points belongs to the domain of the other Segal-chart). A simply calculation leads to:

$$\left.\frac{\partial(x_T^0, x_T^1, x_T^2, x_T^3)}{\partial(x_G^0, x_G^1, x_G^2, x_G^3)}\right|_G = \begin{pmatrix} \dfrac{1+\cos^2\alpha}{2\cos^2\alpha} & 0 & 0 & \dfrac{\sin^2\alpha}{2\cos^2\alpha} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{\sin^2\alpha}{2\cos^2\alpha} & 0 & 0 & \dfrac{1+\cos^2\alpha}{2\cos^2\alpha} \end{pmatrix}.$$

This (linear) transformation can be considered (in two dimensions, with respect to the coordinates x_G^0, x_G^3) as the composition of a dilatation by a factor $\dfrac{1}{\cos\alpha}$ and the following (two-dimensional) *Lorentz transformation*:

$$\frac{1}{2\cos\alpha}\begin{pmatrix} 1+\cos^2\alpha & \sin^2\alpha \\ \sin^2\alpha & 1+\cos^2\alpha \end{pmatrix}.$$

Comparison with the standard Lorentz transformation:

$$\frac{\partial(x_T^0, x_T^3)}{\partial(x_G^0, x_G^3)} = \begin{pmatrix} \delta & \delta\beta \\ \delta\beta & \delta \end{pmatrix}$$

(see example 5 for interpretation) yields the result

$$\beta = \frac{\sin^2\alpha}{1+\cos^2\alpha}, \qquad \delta = \frac{1+\cos^2\alpha}{2\cos\alpha}.$$

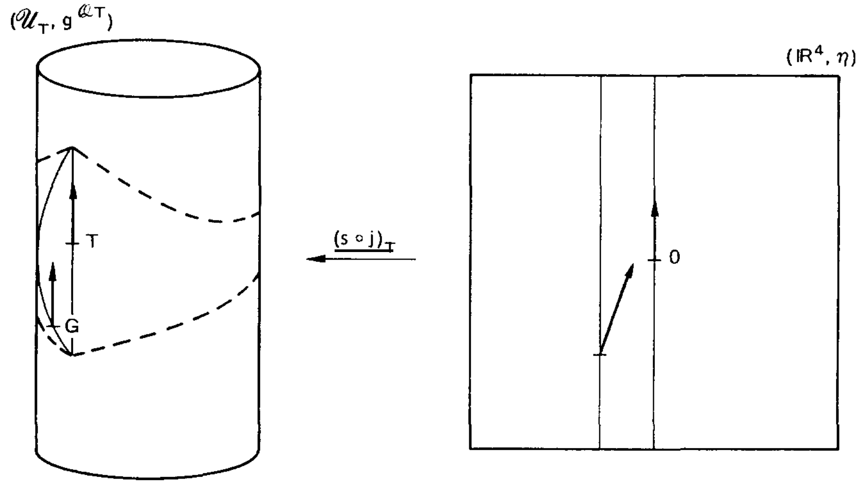
Thus the *observational appearence* of the γ-surrounding region to γ' is characterized by:

- an *enhancement* by a factor $\dfrac{1}{\cos\alpha}$ of the differences between the coordinates (x_T^0, x_T^r) for any pair of events near G, with respect to the corresponding differences of coordinates (x_G^0, x_G^r) ($\equiv$ *contraction of the scales* defining (x_T^0, x_T^r) with respect to the ones defining (x_G^0, x_G^r)).

- a *radial relative approaching* of γ with velocity $\beta_r = \dfrac{-\sin^2\alpha}{1+\cos^2\alpha}$ (because $\beta > 0$ and $x_T^3(G) = -\tan\alpha < 0$); which is shown in the next diagram.

What would be the redshift observed for the radiation emitted by G and received at T?. It is clear that

$$r\begin{pmatrix}\text{enhancement } + \\ \text{approaching}\end{pmatrix} = \frac{1}{\cos\alpha}\cdot r_{\text{approaching}}.$$



But the factor $r_{\text{approching}}$ has been computed in example 5 and the (well known) result is:

$$r_{\text{approaching}} = \delta(1 + \beta_r) = \frac{1 + \cos^2 \alpha}{2 \cos \alpha} \left(1 - \frac{\sin^2 \alpha}{1 + \cos^2 \alpha}\right) = \cos \alpha$$

so it is clear that $r\begin{pmatrix} \text{enhancement} + \\ \text{approaching} \end{pmatrix} = 1$. We see again that *there is no cosmological redshift in Segal's model.*

REMARK. In [2], III-7, Segal develops almost until the end the calculation we have just done, and he gives the expression of the jacobian

$$\left.\frac{\partial(x_T)}{\partial(x_G)}\right|_G .$$

It is very surprising that Segal obtains ([2], III-6)

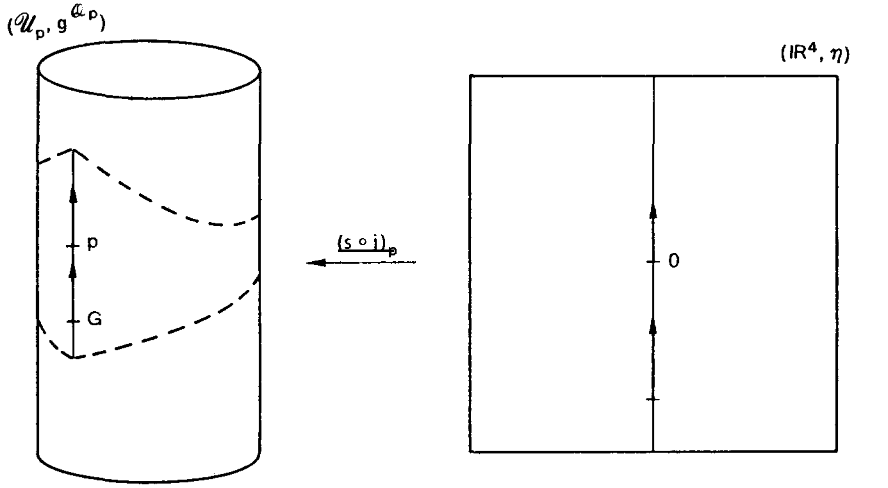
$$r = \frac{2}{1 + \cos \alpha} = 1 + \tan^2 \frac{\alpha}{2} .$$

In out framework *this is equivalent to consider that G and T are on the same integral curve of $Q = \partial_t$. In fact:*

Let us consider the point P on γ with coordinates $t(P) = \alpha (0 < \alpha < \pi)$, $\rho(P) = \vartheta(P) = \varphi(P) = 0$. It holds:

$$\left.\frac{\partial(x_P^0, x_P^1, x_P^2, x_P^3)}{\partial(x_G^0, x_G^1, x_G^2, x_G^3)}\right|_G = \begin{pmatrix} \dfrac{2}{1+\cos\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{2}{1+\cos\alpha} \end{pmatrix}$$

which can be considered (in two dimensions, with respect to the coordinates x_G^0, x_G^3) as a simple dilatation by a factor $\dfrac{2}{1+\cos\alpha}$.



If we accept that r is the one given by this scheme, it is clear that it would follow:

$$r = \frac{2}{1+\cos\alpha}.$$

However, *the redshift is not caused by a «temporal evolution», but rather by a «light propagation»*. This is our main disagreement with Segal (see Section *C*).

C) The third argument is based on a *detailed analysis of Segal's computation* of the redshift in his model. In this «quantum» computation Segal studies the «temporal evolution» of the operator $H_0 := 1/i\ \mathcal{Q}_G \equiv 1/i\,(s\circ j)_{G*}\partial_{x^0_G}$ ($\equiv$ dynamical variable «apparent energy» by means of which, according to Segal, the observer in G analyzes the radiation *he emits*), due to the hamiltonian $H :=$

$= \frac{1}{i}\, Q \equiv \frac{1}{i}\, \partial_t$ (≡ dynamical variable «real energy» in the spacetime of Segal), during an interval $\Delta t = \alpha$ (≡ difference between the values of t at the points G of emission and T of reception). The operator $H_0(\alpha) := e^{-iH\alpha} H_0 e^{+iH\alpha}$ would represent, according to Segal, the dynamical variable «apparent energy» used by the observer in T to analyze the radiation *received at* T and coming from G. Assuming that the wave function of the radiation (plane wave in Segal's approach) is eigenstate of H_0 with eigenvalue ν_E(≡ emission frequency), Segal obtains a reception frequency $\nu_R = \frac{1 + \cos\alpha}{2}\, \nu_E$; therefore $r = \frac{2}{1 + \cos\alpha} = 1 + \tan^2 \alpha/2 > 1$.

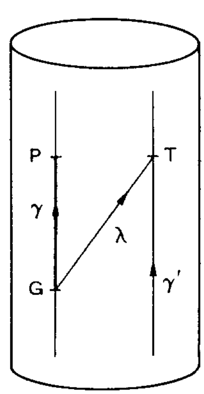
(i) As it was mentioned in the footnote of Section 4, it does not seem that an analysis of the cosmological redshift would need a quantum approach. *We are going to see now that Segal's approach has indeed an immediate «classical» (≡geometrical) analogue*: let us compute the following ratio between scalar produces:

$$R := \frac{\cancel{g^{\partial_t}(\lambda_* 0,\ \mathcal{Q}_G G)}}{g^{\partial_t}(\lambda_* 0, \Pi^{\partial_t}_{PG,\gamma}\ \mathcal{Q}_G P)} = \frac{g^{\partial_t}((\partial_t + \partial_\rho) G,\ \mathcal{Q}_G G)}{g^{\partial_t}((\partial_t + \partial_\rho) G, \Pi^{\partial_t}_{PG,\gamma}\ \mathcal{Q}_G P)}$$

where

- $\lambda : [0, \alpha] \to \mathbb{R} \times S^3$ is the future lightlike geodesic of g^{∂_t} between $\lambda 0 = G$ and $\lambda\alpha = T$, which is an integral curve of the field $(\partial_t + \partial_p)$ (see Remark at the beginning of this section).
- $\Pi^{\partial_t}_{PG,\gamma}$ is the parallel transport defined on γ by the Levi-Civita connection ∇^{∂_t} between the points P and G.

As a consequence of $\mathcal{Q}_G = \left(\frac{1}{2}(1 + \cos t \cos\rho)\, \partial_t - \frac{1}{2} \sin t \sin\rho\, \partial_\rho\right)\Big|_{\mathcal{U}_G}$ (Lemma 22.iv) and $\nabla^{\partial_t}_{\partial_t} \partial_t = 0 = \nabla^{\partial_t}_{\partial_\rho} \partial_t$ (Lemma 13; remember that $Q \equiv \partial_t$ is



temporally homogeneous), it follows:

$$\nabla^{\partial_t}_{\mathcal{Q}_G}\partial_t = 0; \Rightarrow \nabla^{\partial_t}_{\partial_t}\mathcal{Q}_G = [\partial_t, \mathcal{Q}_G]; \Rightarrow$$

$$\Rightarrow \Pi^{\partial_t}_{PG,\gamma}(\mathcal{Q}_G P) = \phi^{\partial_t}_{-\alpha_*}(\mathcal{Q}_G P) \equiv \phi^{\partial_t}_{-\alpha_*} \circ \mathcal{Q}_G \circ \phi^{\partial_t}_{\alpha}(G),$$

$\phi^{\partial_t}_{\alpha}$ being the diffeomorphism induced by the field ∂_t for the value α of the parameter; then we can write (6):

$$\Pi^{\partial_t}_{PG,\gamma}(\mathcal{Q}_G P) = e^{-\alpha\partial_t}\mathcal{Q}_G\, e^{\alpha\partial_t}(G) \equiv e^{-iH\alpha} iH_0 e^{iH\alpha}(G).$$

On the other hand, if we denote by $\mathcal{L}_{ab}$ $(a, b = -1, 0, 2, 3, 4)$ the generators of $\widetilde{SU(2,2)}$ acting causally on $\mathbb{R} \times S^3$, it holds (see Appendix)

$$\partial_t = \mathcal{L}_{-10}, \quad \mathcal{Q}_G \equiv \frac{1}{2}(1 + \cos t \cos \rho)\partial_t - \frac{1}{2}\sin t \sin \rho\, \partial_\rho =$$

$$= \frac{1}{2}(\mathcal{L}_{-10} + \mathcal{L}_{04});$$

thus:

$$\Pi^{\partial_t}_{PG,\gamma}(\mathcal{Q}_G P) = e^{-\alpha\mathcal{L}_{-10}}\frac{1}{2}(\mathcal{L}_{-10} + \mathcal{L}_{04})\, e^{\alpha\mathcal{L}_{-10}}(G);$$

using the commutation relations of the $\mathcal{L}_{ab}$ (see Appendix), one gets:

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(6) Let $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ be an affine diffeomorphism in the standard chart of $\mathbb{R}^n$ and let us denote by δ the map $T\mathbb{R}^n \to \mathbb{R}^n$, $(a, \alpha) \mapsto a + \alpha$. It is very easy to see that the following diagram commutes:

$$\begin{array}{ccc} T\mathbb{R}^n & \xrightarrow{\varphi_*} & T\mathbb{R}^n \\ \delta\downarrow & & \downarrow\delta \\ \mathbb{R}^n & \xrightarrow{\varphi} & \mathbb{R}^n \end{array}$$

Now if φ is the diffeomorphism induced (for a value s of the parameter) by the flow of a vector-field Y with affine coordinate components in the standard chart, we shall have $\varphi = e^{sY}$ (exponential map in the Lie-group of affine trasformation of $\mathbb{R}^n$).

Let X be any vector field on $\mathbb{R}^n$. There exists always a field X' on $\mathbb{R}^n$ such that $\varphi_* X(a) = X'(\varphi a)$, $\forall\, a \in \mathbb{R}^n$, and it holds $\delta \circ X' = \varphi \circ \delta \circ X \circ \varphi^{-1} = e^{sY} \circ \delta \circ X \circ e^{-sY}$; this can be rewritten (with the obvious identification) as $X' = e^{sY} \circ X \circ e^{-sY}$.

This is still true on an arbitrary manifold if we consider the domain of a fixed chart where the above mentioned conditions are satisfied; in our case, let us take the chart $(t, \rho, \vartheta, \varphi)$ on $\mathbb{R} \times S^3$ and let the field Y be either ∂_t or $(\partial_t + \partial_\rho)$.

$$\Pi^{\partial_t}_{PG,\gamma}(\mathcal{Q}_G P) = \frac{1}{2}(\mathcal{L}_{-10} + e^{-\alpha\mathcal{L}_{-10}}\mathcal{L}_{04}e^{\alpha\mathcal{L}_{-10}})(G) =$$

$$= \frac{1}{2}(\mathcal{L}_{-10} - \sin\alpha\,\mathcal{L}_{-14} + \cos\alpha\,\mathcal{L}_{04})(G);$$

finally, taking into account the expressions of the $\mathcal{L}_{ab}$ (see Appendix) and the values $t(G) = u^1(G) = u^2(G) = u^3(G) = 0$, we arrive to

$$\Pi^{\partial_t}_{PG,\gamma}(\mathcal{Q}_G P) = \frac{1}{2}(\partial_t - \sin\alpha \cdot 0 + \cos\alpha \cdot \partial_t)(G) = \frac{1}{2}(1 + \cos\alpha)\partial_t(G)$$

and from here we obtain:

$$R = \frac{g^{\partial_t}((\partial_t + \partial_\rho)G, \partial_t G)}{g^{\partial_t}\left((\partial_t + \partial_\rho)G, \frac{1}{2}(1 + \cos\alpha)\partial_t G\right)} = \frac{2}{1 + \cos\alpha}$$

this is the «classical» version of Segal's computation.

(ii) The comparison between the definition of the factor R in (i) and the expression obtained for the proper time ratio r in the corollary to the Lemma 15

$$r = \frac{g^{\partial_t}(\lambda_* 0, \gamma_* \tau_0)}{g^{\partial_t}(\lambda_* 0, \Pi^{\partial_t}_{TG,\lambda}\gamma'_*(f\tau_0))}$$

shows clearly *two objections* which can be made to the Segal's calculation:

a) As it was mentioned in part B of this section, *the redshift is not related to a time evolution* (≡ transport between points P and G), but rather to a light propagation (≡ transport between points T and G). In quantum mechanical language, *the operator H_0 is not invariant under spatial translations* so that the operator representing the dynacal variable «apparent energy» used by the observer in T to analyze the radiation coming from G will be the result of the «spacetime evolution» of the operator H_0. *This is the main objection to Segal's computation.*

b) *It has no meaning to assume that γ and γ' are integral curves of $\mathcal{Q}$, because they are integral curves of $Q \equiv \partial_t$.* However, this fact *has no incidence* (as we are going to see immediately) in the cosmological redshift, provided the objection (a) has been correctly salved; this is due to the fact that $\mathcal{Q}_G$ is also temporally homogeneous (see Remark in part A of this section).

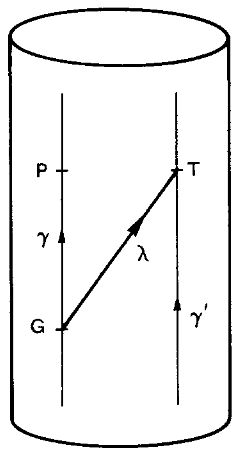
(iii) Here we are going to reform Segal's computation, taking into account the objection (a).

Let us compute (following the scheme of (i)) the ratio between scalar products:

$$R' := \frac{g^{\partial_t}(\lambda_* 0, \mathcal{Q}_G G)}{g^{\partial_t}(\lambda_* 0, \Pi^{\partial_t}_{TG,\lambda} \mathcal{Q}_G T)} = \frac{g^{\partial_t}((\partial_t + \partial_\rho)G, \mathcal{Q}_G G)}{g^{\partial_t}((\partial_t + \partial_\rho)G, \Pi^{\partial_t}_{TG,\lambda} \mathcal{Q}_G T)}$$

with

- λ being as in (i)
- $\Pi^{\partial_t}_{TG,\lambda}$ being the parallel transport defined on λ by the Levi-Civita connection ∇^{∂_t} between the points T and G.



From the arguments given in (i), and because $\nabla^{\partial_t}_{\partial_\rho}\partial_\rho = 0$ (∂_ρ is *geodesic* for ∇^{∂_t}, it's inmediate to check it), it follows:

$$\nabla^{\partial_t}{}_G(\partial_t + \partial_\rho) = 0; \Rightarrow \nabla^{\partial_t}_{(\partial_t + \partial_\rho)} \mathcal{Q}_G = [\partial_t + \partial_\rho, \mathcal{Q}_G]; \Rightarrow$$
$$\Rightarrow \Pi^{\partial_t}_{TG,\lambda}(\mathcal{Q}_G T) = \phi^{\partial_t + \partial_\rho}_{-\alpha_*}(\mathcal{Q}_G T) \equiv \phi^{\partial_t + \partial_\rho}_{-\alpha_*} \circ \mathcal{Q}_G \circ \phi^{\partial_t + \partial_\rho}_{\alpha}(G);$$

then we can write

$$\Pi^{\partial_t}_{TG,\lambda}(\mathcal{Q}_G T) = e^{-\alpha(\partial_t + \partial_\rho)} \mathcal{Q}_G e^{\alpha(\partial_t + \partial_\rho)}(G);$$

taking in account (see Appendix) that:

$$\partial_t = \mathcal{L}_{-10}, \quad \partial_\rho|_{u^1 = u^2 = 0} = \mathcal{L}_{34}, \quad \mathcal{Q}_G = \frac{1}{2}(\mathcal{L}_{-10} + \mathcal{L}_{04}),$$

and using the conmutation relations between the $\mathcal{L}_{ab}$ (see Appendix), one obtains:

$$\Pi^{\partial_t}_{TG,\lambda}(\mathcal{Q}_G T) = \frac{1}{2}(\mathcal{L}_{-10} + e^{-\alpha\mathcal{L}34}(e^{-\alpha\mathcal{L}-10}\mathcal{L}_{04}e^{\alpha\mathcal{L}10})e^{\alpha\mathcal{L}34})(G) =$$

$$= \frac{1}{2}(\mathcal{L}_{-10} - \sin\alpha\, e^{-\alpha\mathcal{L}34}\mathcal{L}_{-14}e^{\alpha\mathcal{L}34} + \cos\alpha\cdot e^{-\alpha\mathcal{L}34}\mathcal{L}_{04}e^{\alpha\mathcal{L}34})(G) =$$

$$= \frac{1}{2}(\mathcal{L}_{-10} - \sin\alpha(\sin\alpha\cdot\mathcal{L}_{-13} + \cos\alpha\cdot\mathcal{L}_{-14}) + \cos\alpha\cdot(\sin\alpha\cdot\mathcal{L}_{03} + \cos\alpha\cdot\mathcal{L}_{04}))(G)$$

finally, taking in account the expressions for the $\mathcal{L}_{ab}$ (see Appendix) and the values $t(G) = u^1(G) = u^2(G) = u^3(G) = 0$, we arrive to:

$$\Pi^{\partial_t}_{TG}(\mathcal{Q}_G T) = \frac{1}{2}(\partial_t - \sin^2\alpha\,\partial_\rho - \sin\alpha\cdot\cos\alpha\cdot 0 + \cos\alpha\cdot\sin\alpha\cdot 0 +$$

$$+ \cos^2\alpha\cdot\partial_t)(G) = \frac{1}{2}((1+\cos^2\alpha)\partial_t - \sin^2\alpha\,\partial_\rho)(G)$$

and from here we obtain:

$$R' = \frac{g^{\partial_t}((\partial_t + \partial_\rho)G, \partial_t G)}{g^{\partial_t}\left((\partial_t + \partial_\rho)G, \frac{1}{2}((1+\cos^2\alpha)\partial_t - \sin^2\alpha\cdot\partial_\rho)(G)\right)} =$$

$$= \frac{-1}{-\frac{1}{2}(1+\cos^2\alpha) - \frac{1}{2}\operatorname{sen}^2\alpha} = 1.$$

(iv) *Conclusion*

The «classical» (≡ geometrical) approach of Segal's computation (evaluation of the factor R) yields the result (see (i)),

$$R := \frac{\cancel{g^{\partial_t}(\lambda_* 0, \mathcal{Q}_G G)}}{g^{\partial_t}(\lambda_* 0, \Pi^{\partial_t}_{PG,\gamma}\mathcal{Q}_G P)} = \frac{2}{1+\cos\alpha};$$

the objection (a) (see (ii)) leads to compute the ratio R' (instead of R); one gets (see (iii)

$$R' := \frac{g^{\partial_t}(\lambda_* 0, \mathcal{Q}_G G)}{g^{\partial_t}(\lambda_* 0, \Pi^{\partial_t}_{TG,\lambda}\mathcal{Q}_G T)} = 1;$$

the objection (b) (see (ii)) leads to compute the ratio r (instead of R'). One obtains (see theorems 1 and 2; the computation is much easier than the ones of R and R'):

$$r := \frac{g^{\partial_t}(\lambda_* 0, \partial_t G)}{g^{\partial_t}(\lambda_* 0, \Pi^{\partial_t}_{TG,\lambda} \partial_t T)} = 1.$$

(The result $r = 1 = R'$ was actually expected; see Remark in part A of this section).

7. APPENDIX

We describe right now the local action of the causal group $\widetilde{SU(2,2)}$ on the Segal's CST $(\mathbb{R} \times S^3, [-dt^2 \oplus h])$. The generators of $O(2,4)$ on $\mathbb{R}^6$ are given by

$$L_{ab} := \epsilon^a w^a \partial_{w^b} - \epsilon^b w^b \partial_{w^a} \quad \left(\text{where } \epsilon^a = \begin{cases} +1, a = -1, 0 \\ \\ -1, a = 1, 2, 3, 4 \end{cases} \right)$$

with Lie brackets $[L_{ab}, L_{bc}] = \epsilon^b L_{ac}$.

The conformal map (Lemma 18) $j : (\mathbb{R}^4, [\eta]) \to (B, \mathscr{C}_B)$ has (on its image, dense in B) an inverse map given by

$$\tilde{w} \mapsto x, \text{ with } x^\mu = \frac{2w^\mu}{(w^{-1} + w^4)} \quad (\mu = 0, 1, 2, 3) \quad [\forall \text{representative element } w \in \tilde{w}]$$

Thus the generators l_{ab} of $SU(2,2)/\mathbb{Z}_4$ acting on $\mathbb{R}^4$ (through j^{-1}) are given by

$$l_{-1\mu} = \left(1 + \frac{\eta(x,x)}{4}\right)\partial_{x^\mu} + \epsilon^\mu \frac{x^\mu}{2} S, \text{ with } S := x^\nu \partial_{x^\nu}$$

$$l_{-14} = -S$$

$$l_{0j} = x^0 \partial_{x^j} + x^j \partial_{x^0}$$

$$l_{ij} = -x^i \partial_{x^j} + x^j \partial_{x^i}$$

$$l_{\mu 4} = \left(1 - \frac{\eta(x,x)}{4}\right)\partial_{x^\mu} - \epsilon^\mu \frac{x^\mu}{2} S.$$

The conformal map (see Remark to Lemma 19) $s : (B, \mathscr{C}_B) \to (\mathbb{R} \times S^3, [-dt^2 \oplus h])$ is given by

$$\begin{cases} t = \arcsin \dfrac{w^0}{(w^{-1^2} + w^{0^2})^{1/2}} \\ u^i = \dfrac{w^i}{(w^{-1^2} + w^{0^2})^{1/2}} \quad (i = 1, 2, 3). \end{cases} \qquad [\forall \text{ representative element } w \in \tilde{w}]$$

Thus the generators $\mathcal{L}_{ab}$ of $SU(2, 2)/\mathbb{Z}_4$ (which are those of $\widetilde{SU}(2, 2)$) acting on $\mathbb{R} \times S^3$ (through s) are given by

$$\mathcal{L}_{-10} = \partial_t$$
$$\mathcal{L}_{-1i} = -u^i \sin t\, \partial_t + \cos t(\partial_u i - u^i \Sigma), \quad \text{with } \Sigma := u^j \partial_u j$$
$$\mathcal{L}_{-14} = -(1 - u^{1^2} - u^{2^2} - u^{3^2})^{1/2}(\sin t\, \partial_t + \cos t\, \Sigma)$$
$$\mathcal{L}_{0j} = \sin t\, \partial_u j + u^j(\cos t\, \partial_t - \sin t\, \Sigma)$$
$$\mathcal{L}_{ij} = -u^i \partial_u j + u^j \partial_u i$$
$$\mathcal{L}_{04} = (1 - u^{1^2} - u^{2^2} - u^{3^2})^{1/2}(\cos t\, \partial_t - \sin t\, \Sigma)$$
$$\mathcal{L}_{i4} = (1 - u^{1^2} - u^{2^2} - u^{3^2})^{1/2} \partial_u i.$$

We finally explain some notations concerning different kinds of coordinates used in the text:

On $S^1 \times S^3$

$$(u^{-1}, u^0, u^1, u^2, u^3, u^4) \quad \text{satisfying} \begin{cases} u^{-1^2} + u^{0^2} = 1 \\ u^{1^2} + u^{2^2} + u^{3^2} + u^{4^2} = 1 \end{cases}$$

$$\text{change} \begin{cases} u^{-1} = \cos t \\ u^0 = \sin t \end{cases} \quad \Big\downarrow$$

[also on $\mathbb{R} \times S^3$] (t, u^1, u^2, u^3, u^4)

$$\text{change} \begin{cases} u^1 = \sin\rho \sin\vartheta \sin\varphi \\ u^2 = \sin\rho \sin\vartheta \cos\varphi \\ u^3 = \sin\rho \cos\vartheta \\ u^4 = \cos\rho \end{cases} \quad \Big\downarrow$$

[also on $\mathbb{R} \times S^3$] $(t, \rho, \vartheta, \varphi)$

On $\mathbb{R}^4$

$$(x^0, x^1, x^2, x^3)$$

$$\text{change} \begin{cases} x^1 = x^r \sin x^\vartheta \sin x^\varphi \\ x^2 = x^r \sin x^\vartheta \cos x^\varphi \\ x^3 = x^r \cos x^\vartheta \end{cases} \Big\downarrow$$

$$(x^0, x^r, x^\vartheta, x^\varphi).$$

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